QFT & HEP

Abstract

I don't really know how to describe this notes, they are just a supplement to the main notes but well organised in my opinion and with multiple inputs from other books. Very few formulas will be derived, however the goal is to try to explain the physics of all of them.

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Notation

Natural units	$c = \hbar = 1$
Minkowski Metric	$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$
4-vector	$x^\mu = (x_0, \boldsymbol{x})$
4-covector	$x_\mu = \eta_{\mu\nu} x^\nu = (x_0, -\boldsymbol{x})$
4-derivative	$\partial_\mu \phi = (\partial_t \phi, \boldsymbol{\nabla} \phi)$
D'Alembertian	$\Box = \partial^2 = \partial^\mu \partial_\mu = \partial_t^2 - \nabla^2$
Relativistic energy	$E_{\boldsymbol{p}} = \sqrt{\lvert\boldsymbol{p}\rvert^2 + m^2}$
Barn	$1\,\mathrm{barn} = 1 \times 10^{-24}\,\mathrm{cm}^2$
Normal order	$\mathbf{N}\left[\phi_1 \phi_2\right]$
Time order	$\mathbf{T}\left\{\phi_1 \phi_2\right\}$

Why a Quantum Field Theory?

Of course, the inadequacies of classical physics have been known for more than a hundred years by now. The classical theories we observe in nature actually arise as "approximations" or "limits" of quantum theories. I can't give a general definition of a quantum field theory (no-one can, at least not a satisfactory definition), but I can describe some properties these theories must have [Ell13].

1. An important characteristic of quantum theory is the nature of measurement: what kind of thing is a quantum observation? Here's a sign that something genuinely different is going on to the classical theory: the observations we can make with "true" or "false" as possible answers fail to form a Boolean algebra. The famous counterexample is Young's two slit experiment. Suppose one has a screen with two slits at points A and B, and a detector at a point C beyond it, and one fires a single photon at the screen. Then one can do two different experiments, measuring two different possible observables. One finds different results by performing the following two measurements:

   $$(A \text{ OR } B) \text{ AND } C = (A \text{ AND } C) \text{ OR } (B \text{ AND } C)$$

   where by A, B, C I mean the observables "was a particle detected at this point?" There are two things to observe here. The first is the failure of the distributivity law (as satisfied by measurements in classical mechanics), the second is the non-determinism of the situation: one generally doesn't get the same result when one repeats the same experiment. Quantum measurements are inherently probabilistic. As a result, while we cannot meaningfully talk about the value of an observable when the system is in some state, it does make sense to talk about the expected value of an observable.

2. Another famous characteristic that our model for quantum observables must possess is failure of simultaneous measurability. This is typified by Heisenberg's uncertainty principle: two observable quantities for a quantum particle are its position and its momentum. Suppose one tried to build an algebra of observables, where the product was "do both observables simultaneously". Measuring position and momentum simultaneously should certainly arise as a limit of "measure position, then measure momentum time later" as $\epsilon \to 0$, or likewise of "measure momentum, then measure position time ϵ later". The uncertainty principle tells us that in fact these limits necessarily differ. While one can produce an algebra of observables, it is necessarily non-commutative in all non-trivial examples.

3. I should say something about the quantum notion of "states", and the wave-particle duality in quantum mechanics. One wants to represent our algebra of observables as acting on something. The principle of superposition says that any complex linear combination of two quantum states is also a state (as in the thought experiment of Schrödinger's cat, but in fact this is an experimentally verifiable phenomenon), so our space of states forms a complex vector space. One generally thinks of the space of states as a separable Hilbert space, with the observables acting by self-adjoint operators.

   For example, in the case of a quantum particle moving in $\mathbb{R}^n$, we have the position and momentum operators, which satisfy well-known commutation relations. The Stone-von Neumann theorem tells us that the representation of these operators is essentially unique, and can be described as multiplication and differentiation operators acting on the Hilbert space $L^2(\mathbb{R}^n)$

4. I've mostly spoken just about quantum mechanics. In quantum field theory we really need to remember a piece of data we've been so far essentially forgetting: the underlying spacetime manifold. When we consider observables in this context we can remember the data of the support of a classical observable: does it only depend on a field in a certain neighbourhood? Quantisation should reflect this locality in a suitable way (I won't discuss this further, because it's somewhat orthogonal to the rest of the talk, but models for quantum field theory, both descriptions like TQFTs and descriptions like factorisation algebras have this locality built in as a hypothesis).

5. Finally, our system must behave well in the classical limit. That is, if we take a limit at low energies, or at long distances, we should recover the appropriate classical field theory. What does this mean in terms of observables? Well, broadly speaking, to any quantum observable there should correspond an underlying classical observable – a function on the classical state space – and the commutator of quantum observables should agree with the Poisson bracket of the classical observables up to a factor of $i\hbar$.

1 Classical Field Theory

The dynamics of a system is encoded in a Lagrangian over the continuous fields $\phi_i = \phi_i(x^\mu)$ by the Lagrangian density function

$$\mathcal{L} = \mathcal{L}(\phi_i; \partial_\mu \phi_i) \tag{1.1}$$

The Lagrangian L is obtained from the Lagrangian density $\mathcal{L}$ by integrating over the space variables

$$\mathrm{L} = \int d^3x \ \mathcal{L}(\phi_i; \partial_\mu \phi_i) \tag{1.2}$$

Properties

1. Local: each product is evaluated at the same space-time point, a term like $\phi_i(x)\phi_j(x')$ is not valid.
2. Lorentz invariant: the whole Lagrangian has to be Lorentz invariant, i.e., it should not depend on the reference frame.
3. Real: $\mathcal{L} \in \mathbb{R}$ (although it might not be explicit).

Define the action $S(\Omega)$ as the integral over an arbitrary region of the 4-d space time continuum of the Lagrangian density

$$S(\Omega) = \int_\Omega d^4x \ \mathcal{L}(\phi_i; \partial_\mu \phi_i) \tag{1.3}$$

Just like in classical mechanics, the equations of motion are obtained from the variational principle.We consider variations of the fields $\phi_i(x) \to \phi_i(x) + \delta\phi_i(x)$, with such variations that vanish at the surface of Ω. Then, at first order, the equations of motion are obtained imposing that the action is stationary, $\delta S = 0$, which gives

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi_i]}\right) = 0 \tag{1.4}$$

for each field ϕ_i with $i = 1, \ldots, N$.

Define the canonical conjugate momenta

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} \tag{1.5}$$

and with this the Hamiltonian density

$$\mathcal{H}(\phi_i; \pi_i; \boldsymbol{\nabla}\phi_i) = \sum_i \pi_i \dot{\phi}_i - \mathcal{L}(\phi_i; \partial_\mu \phi_i) \tag{1.6}$$

Similarly, the total Hamiltonian is the integral over the space part of the Hamiltonian density

$$\mathrm{H} = \int d^3x \ \mathcal{H}(\phi_i; \pi_i; \boldsymbol{\nabla}\phi_i) \tag{1.7}$$

1.1 Quantization

In order to go from the classical to the quantum field theory, we need to impose some commutation relations to quantize the fields through the Poisson brackets. Let $F[\phi, \pi]$ and $G[\phi, \pi]$ be two functionals of ϕ and π, the *equal time* Poisson brackets are

$$[F, G]_{PB} = d^3x \ \left(\frac{\partial F}{\partial \phi(x)}\frac{\partial G}{\partial \pi(x)} - \frac{\partial F}{\partial \phi(x)}\frac{\partial G}{\partial \pi(x)}\right) \tag{1.8}$$

which is related to the usual commutator in Quantum Physics by

$$[-, -]_Q \longrightarrow i\hbar[-, -]_{PB} \tag{1.9}$$

We can interpret the fields and conjugate momenta as Heisenberg operators (evolving with time) and obtain the canonical commutation relations

$$\begin{aligned} &[\phi_i(t, \boldsymbol{x}), \pi_j(t, \boldsymbol{y})] = i\hbar\delta_{ij}\delta^3(\boldsymbol{x} - \boldsymbol{y}) \\ &[\phi_i(t, \boldsymbol{x}), \phi_j(t, \boldsymbol{y})] = [\pi_i(t, \boldsymbol{x}), \pi_j(t, \boldsymbol{y})] = 0 \end{aligned} \tag{1.10}$$

1.2 Symmetries and conservation laws

Following from Noether's theorem (see Appendix B), with each symmetry the exist a conservation law $\delta\mathcal{L} = 0$ and a current that is conserved of the form

$$\partial_\mu j^\mu = 0 \tag{1.11}$$

which implies the existence of a conserved charge $Q = \int_V d^3x\ j^0$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\int_{\partial V} \boldsymbol{j}\cdot d\boldsymbol{\sigma} \longrightarrow 0 \tag{1.12}$$

Where we assume that the current tends to zero much faster than the increase of S at infinity.

In general, this happens when the change in the Lagrangian is a total derivative, i.e. $\mathcal{L}' = \mathcal{L} + \partial_\alpha \Lambda^\alpha$.

Internal symmetries Those are related to an infinitesimal change in the field of the type*

$$\phi_i(x) \longrightarrow \phi_i'(x) = \phi_i(x) + \delta\phi_i(x) \tag{1.13}$$

The change in the Lagrangian is

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial[\partial_\mu\phi_i]}\delta\phi_i \right) = \partial_\mu(\pi_i^\mu \delta\phi_i) \tag{1.14}$$

and the conserved current, imposing $\delta\mathcal{L} = 0$, according to eq. (1.11) is

$$j_i^\mu = \pi_i^\mu \delta\phi_i \tag{1.15}$$

Space-time symmetries A consequence of a change in the coordinates like

$$\phi_i(x^\mu) \longrightarrow \phi_i(x^\mu + \epsilon^\mu) = \phi_i(x^\mu) + \epsilon^\mu \partial_\mu \phi_i(x^\mu) \tag{1.16}$$

Induces 4 conserved currents (for each field) expressed in terms of the energy-momentum tensor

$$T_i^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial[\partial_\mu\phi_i]}\partial^\nu\phi_i - \eta^{\mu\nu}\mathcal{L} \tag{1.17}$$

such that

$$\partial_\mu T_i^{\mu\nu} = 0 \tag{1.18}$$

Correspondingly, there are 4 conserved charges given by

$$P^\mu = \int d^3x\ T^{\mu 0} \tag{1.19}$$

with $P^0 = \mathrm{H}$.

*This is true if the change is continuous, for example, a discrete change may be $\phi(x) \to -\phi(x)$ which does NOT induce a conserved current.

2 Collisions in Special Relativity

Teorema 1. *The 4-momentum of the whole system is always conserved in a relativistic collision.*

The previous theorem condenses the conservation of energy and momentum in one simple equation. If $\{p_i^\mu\}$ $i \in \mathcal{I}$ is the collection of all 4-momentum of the initial states and $\{p_f^\mu\}$ $f \in \mathcal{F}$ the collection of states after the collision. Then, they must verify that

$$\sum_{i\in\mathcal{I}} p_i^\mu = \sum_{f\in\mathcal{F}} p_f^\mu \tag{2.1}$$

Like in classical collisions, we can still distinguish between elastic and inelastic ones. The first, implies that the particles in the initial state are equal to the final state $\mathcal{I} = \mathcal{F}$ while in the later, obviously, the particles in the initial and final state are different $\mathcal{I} \neq \mathcal{F}$.

We will restrict ourselves in this chapter to study in detail on type of collision involving 4 particles in total: $1 + 2 \longrightarrow 3 + 4$. The conservation of 4-momentum eq. (2.1) implies*

$$p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu \tag{2.2}$$

Although the last equation is valid in all reference frames, it is usually studied in two frames of references:

a) LAB: we take the second particle (particle B in our case) to be at rest, $\boldsymbol{p}_B^{LAB} = \mathbf{0}$ so $E_B^{LAB} = m_B$, and the momentum of the other particles are written with respect to that.

b) CM: we situate ourselves in a frame where we see the two initial particles come to us with the same 3-momentum (in modulus) and opposite direction, $\boldsymbol{p}_A^{CM} = -\boldsymbol{p}_B^{CM}$.

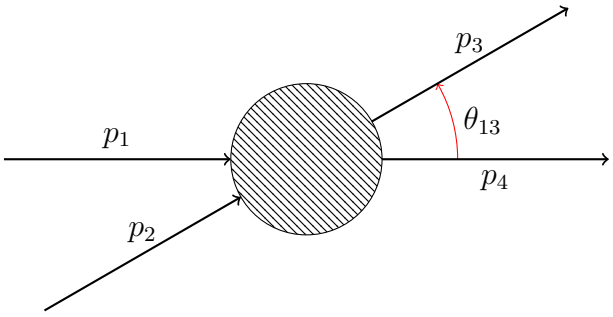


Figure 2.1: Collision between two particles, the shaded area corresponds to the interaction area.

2.1 Mandelstam variables

This are a set of 3 Lorentz invariant quantities (s, t, u) that encode the energy, momentum and angles of the scattering processes like eq. (2.2). They are defined as

$$s = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2p_1p_2 \tag{2.3a}$$
$$t = (p_1 - p_3)^2 = m_1^2 + m_3^2 - 2p_1p_3 \tag{2.3b}$$
$$u = (p_1 - p_4)^2 = m_1^2 + m_4^2 - 2p_1p_4 \tag{2.3c}$$

which verify

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \tag{2.4}$$

These are important because they represent Lorentz invariant quantities, i.e. their value is the same in all RF. Therefore, if we can express the energy and 3-momentum of the particles in terms of Lorentz invariant quantities $s, t, u, m_i...$ we would have solved the problem in all RF as we can always apply a Lorentz transformation to change between reference frames.

*Usually the superscript with the indices μ is omitted and simply writen $p_1 + p_2 = p_3 + p_4$.

Teorema 2. *To define completely the final state, for collisions involving N particles, they are necessary*

$$\begin{cases} 3N-10 & N \geq 4 \\ 0 & N = 3 \end{cases}$$

kinematic variables.

Proof. If we consider N particles, in general we would need $4N$ independent variables but there are some constrains between them. First, we can subtract N because particles are on-shell ($p^2 = E^2 - m^2$), 4 more by conservation of 4-momentum eq. (2.2), 3 if we set one particle to be at rest $\boldsymbol{p}_i = \mathbf{0}$, 2 by changing the axis so that one particle lays completely on the z-axis and finally, 1 more by imposing that another particle is in the y plane so $\boldsymbol{p}_j = (p_{j,x}, 0, p_{j,z})$. If we do this calculation we obtain $3N-10$. However, this equation doesn't make sense for $N = 3$ as we cannot impose the last condition, we only have $3 \times 3 - 9 = 0$. □

Note that, for $N = 4$, from the previous formula we deduce that we only need 2 variables (t, u) for the final state, so the initial state must be defined completely by s. It is instructive to work out the relation between s and the energies and momentum of the initial state in the two RF that we will use.

There is one last quantity that is usually calculated for the type of collisions show in fig. 2.1 as it is easy to measure experimentally.We refer to the angle θ between particles 1 and 3, which can be derived using the t Mandlestam variable

$$\cos\theta_{13} = \frac{t - m_1^2 - m_3^2 + 2E_1E_3}{2|\boldsymbol{p}_1||\boldsymbol{p}_3|} \tag{2.5}$$

LAB frame Remember, in here the second particle is at rest $\boldsymbol{p}_2^{LAB} = 0 \Rightarrow E_2^{LAB} = m_2$ and the first particle collides with it with momentum $p_1^{LAB} = (E_1^{LAB}, \boldsymbol{p}_1^{LAB})$. The only constraint that we have is imposed by Einstein's relation and asserts that

$$p^2 = E^2 - |\boldsymbol{p}|^2 = m^2 \tag{2.6}$$

From the definition of s eq. (2.3a), with the values of p_2^{LAB}, we are left with $s = m_1^2 + m_2^2 + 2E_1^{LAB}m_2$ from which we can isolate the energy of the first particle in terms of s and the masses of the particles

$$E_1^{LAB} = \frac{s - m_1^2 - m_2^2}{2m_2} \tag{2.7}$$

and using Einstein's relation eq. (2.6), we have the expression for the modulus of the momentum of the first particle in the LAB frame in terms of the same variables

$$\left|\boldsymbol{p}_1^{LAB}\right| = \sqrt{\left(E_1^{LAB}\right)^2 - m_1^2} = \frac{1}{2m_2}\lambda^{1/2}(s, m_1^2, m_2^2) \tag{2.8}$$

where

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz = \left[x^2 - (\sqrt{y} + \sqrt{z})^2\right]\left[x^2 - (\sqrt{y} - \sqrt{z})^2\right] \tag{2.9}$$

Then, the total energy is of course

$$E_T^{LAB} = E_1^{LAB} + m_2 = \frac{s - m_1^2 + m_2^2}{2m_2} \tag{2.10}$$

CM frame The derivation in this case is not so easy. First of all, the momentum of the initial particles move in the same direction but with opposite signs so $\boldsymbol{p}_1^{CM} = -\boldsymbol{p}_2^{CM} \Rightarrow \left|\boldsymbol{p}_1^{CM}\right| = \left|\boldsymbol{p}_2^{CM}\right| = \left|\boldsymbol{p}^{CM}\right|$. The expression of the Mandlestam variable s, using the previous relations, is

$$s = m_1^2 + m_2^2 + 2p_1^{CM}p_2^{CM} = m_1^2 + m_2^2 + 2\left(E_1^{CM}E_2^{CM} + \left|\boldsymbol{p}^{CM}\right|^2\right) \tag{2.11}$$

where $\left|\boldsymbol{p}^{CM}\right|^2 = \left(E_1^{CM}\right)^2 - m_1^2 = \left(E_2^{CM}\right)^2 - m_2^2$. Equating both expressions, it follows that

$$s = \left(E_1^{CM} + E_2^{CM}\right)^2 \Longrightarrow E_T^{CM} = E_1^{CM} + E_2^{CM} = \sqrt{s} \tag{2.12}$$

Substituting in the previous expression we obtain

$$E_1^{CM} = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \quad , \quad E_2^{CM} = \frac{s - m_1^2 + m_2^2}{2\sqrt{s}} \tag{2.13}$$

Finally, the expression for the modulus of the momentum is obtained with Einstein's relation and can be written in terms of the λ function defined before in eq. (2.9) as

$$\left|\boldsymbol{p}^{CM}\right| = \left|\boldsymbol{p}_1^{CM}\right| = \left|\boldsymbol{p}_2^{CM}\right| = \frac{1}{\sqrt{s}}\lambda^{1/2}(s, m_1^2, m_2^2) \tag{2.14}$$

Using the later results, together with the expression for θ_{13} in eq. (2.5) we find the expression for this angle in the CM frame

$$\cos\theta_{13}^{CM} = \frac{s(t-u) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\lambda^{1/2}(s, m_1^2, m_2^2)\lambda^{1/2}(s, m_3^2, m_4^2)} \tag{2.15}$$

There are special cases where the expression for this angle simplifies, for example:

- Elastic collision: $m_1 = m_3$ and $m_2 = m_4$

$$\cos\theta_{13}^{CM} = \frac{s^2 + 2s(t - m_1^2 - m_2^2) + (m_1^2 - m_2^2)^2}{\lambda(s, m_1^2, m_2^2)} \tag{2.16}$$

- Compton scattering: $m_1 = m_3 = 0$ and $m_2 = m_4 = m \neq 0$

$$\cos\theta_{13}^{CM} = \frac{s^2 + 2s(t - m_2^2) + m_2^4}{\lambda(s, 0, m_2^2)} \tag{2.17}$$

- Equal masses: $m_1 = m_2 = m_3 = m_4 = m$

$$\cos\theta_{13}^{CM} = \frac{s^2 + 2s(t - 2m^2)}{\lambda(s, m^2, m^2)} = 1 + \frac{2t}{s(s^2 - 4m^2)} \tag{2.18}$$

On the CM frame It is feasible to ask in which cases we can find a boost that transforms our frame to the centre of mass frame. Consider a system of N particles in some Lorentz frame with masses $m_i \geq 0$ and arbitrary p_i. We want that $\boldsymbol{p}_T^{CM} = \boldsymbol{0}$.

Suppose $\boldsymbol{p}'_T$ is the total momentum in the original frame, we can always apply a rotation so that it lays completely on the x axis: $\boldsymbol{p}_T = R\boldsymbol{p}'_T = p_x\hat{x}$. Then, the Lorentz transformation implies that

$$p_x^{CM} = \gamma(p_x - v_x E) \Longrightarrow v_x = \frac{p_x}{E}$$

but v_x is restricted to be less than the velocity of light so

$$v_x < 1 \Longrightarrow p_x < E$$

Once, in the centre of mass frame, we can perform a rotation to return to the original axis as the centre doesn't suffer any change. Thus, can we conclude that it is always possible to go to the CM frame? It will be the case if $|\boldsymbol{p}| < E$, by definition

$$|\boldsymbol{p}| = \left|\sum_i \boldsymbol{p}_i\right| \leq \sum_i |\boldsymbol{p}_i| = \sum_i \sqrt{m_i^2 + |\boldsymbol{p}_i|^2} = E$$

The important part here is what happens when the inequality converts to an equality. This will be the case of massless particles travelling with parallel momentum $\boldsymbol{p}_i \parallel \boldsymbol{p}_j \ \forall i, j$ then $|\boldsymbol{p}| = E$ and $v_x = 1$ and it won't be possible to go to the CM frame.

In any other case, if only one of the particles is massive, then it is possible to find a boost that relates both systems.

Dalitz plot The definition of θ_{13} offers us a useful tool to impose some restrictions between the s, t, u variables and the masses of the particles. For example, in the later case, where all the masses are the same, $s = 4E^2 \geq 4m^2 \geq 0$ but we know that $\cos\theta_{13}^{CM} \in [-1, 1]$, therefore the second term must be negative in eq. (2.18) so $t \leq 0$. We can perform this analysis for the three variables and for each one we obtain the constraints

$$s = 4E^2 \geq 4m^2 \tag{2.19a}$$

$$t = -2\left|\boldsymbol{p}^{CM}\right|^2(1 - \cos\theta_{13}^{CM}) \leq 0 \tag{2.19b}$$

$$u = -2\left|\boldsymbol{p}^{CM}\right|^2(1 + \cos\theta_{13}^{CM}) \leq 0 \tag{2.19c}$$

which can be represented in a Dalitz plot, see Figure 2.2.

There exist a useful relation concerning the product of the 3 variables s, t, u that states

$$stu \geq as + bt + cu \tag{2.20}$$

where

$$\begin{aligned} a &= (m_1m_2 - m_3m_4)(m_1 + m_2 - m_3m_4)/m_T \\ b &= (m_1m_3 - m_2m_4)(m_1 + m_3 - m_2m_4)/m_T \\ c &= (m_1m_4 - m_2m_3)(m_1 + m_4 - m_2m_3)/m_T \end{aligned}$$

and we have defined the total mass $m_T = \sum_{i=1}^{4} m_i$.

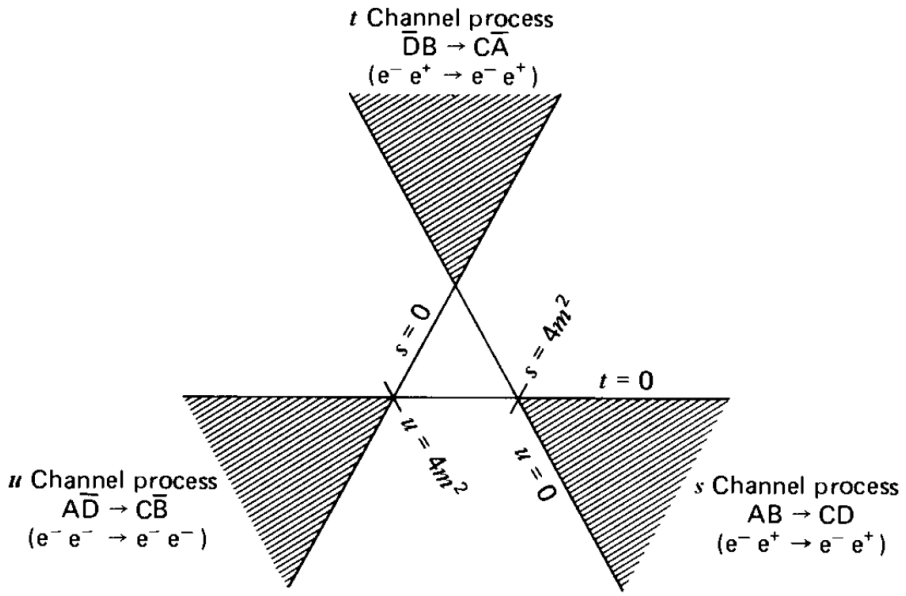


Figure 2.2: The Dalitz plot showing the physical regions (shaded areas) for $e^-e^+ \to e^-e^+$ and the crossed reactions. For scattering between particles of unequal masses, the boundaries of the physical regions are more complicated, but the general result of three non-overlapping regions holds true (See [HM84], Figure 4.7).

Crossing symmetry This plot contains all the information about a scattering process, in concrete the case for $e^-e^+ \to e^-e^+$, with the allowed values that the Mandlestam variables can take. But not only this, it also contains information about the crossed processes, instead of $AB \to CD$ we can also have $A\overline{D} \to C\overline{B}$ and $\overline{D}B \to C\overline{A}$.

The crossing symmetry tells us that, in any interaction, the initial and final particles can be interchanged just by turning them into its antiparticle but the functions describing the two processes are the same.*

*See http://bolvan.ph.utexas.edu/~vadim/Classes/11f/crossing.pdf for a more detailed explanation.

For example, consider the Compton scattering process $e^-\gamma \to e^-\gamma$ where $\mathscr{M}_{compton} \sim F(s,t,u)$ is the probability amplitude for this process. Now consider the complimentary process where we exchange the final electron with the initial photon, the reaction is $e^-e^+ \to \gamma\gamma$ by virtue of the crossing symmetry. The amplitude of this process is $\mathscr{M}_{anhiltation} \sim F(t,s,u)$, the function has not change, only the position of the variables.

Let's see this, in the first interaction we can label the four 4-momentums associated to it as: $p_1 = p_i(e^-)$, $p_2 = p_i(\gamma)$, $p_3 = p_f(e^-)$ and $p_4 = p_f(\gamma)$; Thus, the conservation of momentum reads

$$p_i(e^-) + p_i(\gamma) = p_f(e^-) + p_f(\gamma) \tag{2.21}$$

which can be expressed as

$$p_i(e^-) - p_f(e^-) = p_f(\gamma) - p_i(\gamma) \tag{2.22}$$

However, for the second process we can label the states as $p_1 = p_i(e^-)$, $p_2 = p_i(e^+)$, $p_3 = p_f(\gamma)$ and $p_4 = p_f(\gamma)$; so the conservation of 4-momentum implies

$$p_i(e^-) + p_i(e^+) = p_f(\gamma) + p_f(\gamma) \tag{2.23}$$

In comparison with the previous expression, we identify $p_i(e^+) \leftrightarrow -p_f(e^-)$ and $p_f(\gamma) \leftrightarrow -p_i(\gamma)$. Notice that, the momentum of the antiparticle is related to the momentum of the particle by a change of sign,

$$p(\overline{\pi}) = -p(\pi) \tag{2.24}$$

This conclusion follows directly from the QFT derived in the next sections.

2.2 Decays

A decay of a particle A is a process in which this particle, by itself, is annihilated and turned into some other particles $B, C, \ldots$. We will study the process in which

$$A \longrightarrow \overline{B}CD \tag{2.25}$$

We do so because of the crossing symmetry derived before, in this manner, this process is physically identical to eq. (2.2), we just have to rename the variables.

It is easy to study a decay from the CM frame, considering the mother particle A at rest, $P = (M, \mathbf{0})$ so the conservation laws read

$$\mathbf{0} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 \tag{2.26a}$$

$$M = E_1 + E_2 + E_3 \tag{2.26b}$$

We can use the definition of the Mandlestam variables eq. (2.3) to find an expression for the energies in terms of Lorentz invariant quantities. The only trick here is in the definition of s, previously we had $s = (P + p_1)^2$ but we exchanged p_1 for its antiparticle so by eq. (2.24) we conclude that $s = (P - p_1)^2$. Then, the expressions for the energies is

$$E_1 = \frac{1}{2M}(M^2 + m_1^2 - m_{23}^2) \tag{2.27a}$$

$$E_2 = \frac{1}{2M}(M^2 + m_2^2 - m_{13}^2) \tag{2.27b}$$

$$E_3 = \frac{1}{2M}(M^2 + m_3^2 - m_{12}^2) \tag{2.27c}$$

where we have defined the invariant quantities

$$m_{ij}^2 = (p_i + p_j)^2 \tag{2.28}$$

The magnitude of the final momenta for each particle is

$$|\boldsymbol{p}_i| = \frac{1}{2M}\lambda^{1/2}(m_{jk}^2, M^2, m_i^2) \tag{2.29}$$

where the indices j, k are those different from i.

3 The Klein-Gordon field

From the relativistic equation of the energy

$$E_{\boldsymbol{p}}^2 = m^2 + \boldsymbol{p}^2 \tag{3.1}$$

the idea is to interpret E and $\boldsymbol{p}$ as operators in quantum physics. Then, replacing $\boldsymbol{p} \to -i\boldsymbol{\nabla}$ and $E \to i\partial_t$ leads to

$$(\partial^2 + m^2)\phi(x) = 0 \tag{3.2}$$

This is the Klein-Gordon equation, only valid for particles with spin 0.

The Klein-Gordon equation (eq. (3.2)) can be derived from the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \tag{3.3}$$

and the conjugate momenta is $\pi = \dot{\phi}$. Thus, the Hamiltonian for the KG field is

$$\mathrm{H} = \frac{1}{2}\int d^3x \ \left[\pi^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right] \tag{3.4}$$

and the 3-momentum

$$\boldsymbol{P} = -\int d^3x \ \dot{\phi}\boldsymbol{\nabla}\phi \tag{3.5}$$

The solutions to eq. (3.2) have the form of plane waves

$$\phi^{\pm}(x) \propto e^{\mp p_\mu x^\mu} \tag{3.6}$$

with the condition $p^2 = m^2$ that follows from the KG equation.

3.1 Quantization

The next step is to quantize the field by imposing the canonical commutation relations eq. (1.10)

$$\begin{aligned}[\phi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] &= i\hbar\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\ [\phi(t,\boldsymbol{x}), \phi(t,\boldsymbol{y})] &= [\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] = 0\end{aligned} \tag{3.7}$$

A general solution will be a combination of this plane waves, for all allowed values of k^μ. Suppose, for now, that the field is real ($\phi = \phi^\dagger$), then it can be written as

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_{\boldsymbol{p}}}}\left[a(\boldsymbol{p})e^{-ipx} + a^\dagger(\boldsymbol{p})e^{ipx}\right] = \phi^+(x) + \phi^-(x) \tag{3.8}$$

where

$$\phi^+(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_{\boldsymbol{p}}}} a(\boldsymbol{p})e^{-ipx} \tag{3.9a}$$

$$\phi^-(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_{\boldsymbol{p}}}} a^\dagger(\boldsymbol{p})e^{+ipx} \tag{3.9b}$$

Equation eq. (3.8) makes explicit the dual particle and wave interpretations of the quantum field $\phi(x)$. On the one hand, $\phi(x)$ is written as a Hilbert space operator, which creates and destroys the particles that are the quanta of field excitation. On the other hand, $\phi(x)$ is written as a linear combination of solutions (e^{ipx} and e^{-ipx}) of the Klein-Gordon equation. Both signs of the time dependence in the exponential appear, although $p_0 = E_{\boldsymbol{p}}$ is always positive. If these were single-particle wavefunctions, they would correspond to states of positive and negative energy; let us refer to them more generally as positive- and negative-frequency modes. The connection between the particle creation operators and the waveforms displayed here is always valid for free quantum fields: a positive-frequency solution of the field equation has as its coefficient the operator that

destroys a particle in that single-particle wavefunction. A negative-frequency solution of the field equation, being the hermitian conjugate of a positive-frequency solution, has as its coefficient the operator that creates a particle in that positive-energy single-particle wavefunction. In this way, the fact that relativistic wave equations have both positive- and negative-frequency solutions is reconciled with the requirement that a sensible quantum theory contain only positive excitation energies [PS95].

In fact, eq. (3.8) can be seen as the Fourier transform of the field, from the space-time coordinates to the momentum space. So, we can also write the canonical commutation relations in Fourier space, inserting eq. (3.8) into eq. (3.7) we obtain

$$\begin{aligned}[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] &= \delta^3(\boldsymbol{p}-\boldsymbol{p}') \\ [a(\boldsymbol{p}), a(\boldsymbol{p}')] &= [a^\dagger(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = 0\end{aligned} \tag{3.10}$$

Completely equivalent to the harmonic oscillator commutation relations (see Appendix C). Therefore, in complete analogy, we can interpret $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ as the annihilation and creation operators for a particle with momentum $\boldsymbol{p}$. We may define the vacuum state of the Klein-Gordon field as

$$a(\boldsymbol{p})\,|0\rangle = 0 \quad \forall \boldsymbol{p} \tag{3.11}$$

and by continuously applying different $a^\dagger(\boldsymbol{p})$ to the vacuum state we can populate the field. A state with n particles is expressed as

$$|\boldsymbol{p}_1;\boldsymbol{p}_2;\ldots;\boldsymbol{p}_n\rangle = a^\dagger(\boldsymbol{p}_1)a^\dagger(\boldsymbol{p}_2)\cdots a^\dagger(\boldsymbol{p}_n)\,|0\rangle \tag{3.12}$$

Similarly, this two equations can be expressed using the field operators in eq. (3.8)

$$\phi^+(x)\,|0\rangle = 0 \quad \forall x \tag{3.13}$$

which is clearly understood, as the field destroys the vacuum at all points in space-time.

Since the creation operators commute we have that

$$|\boldsymbol{p}_1;\ldots;\boldsymbol{p}_i;\ldots;\boldsymbol{p}_j;\ldots;\boldsymbol{p}_n\rangle = |\boldsymbol{p}_1;\ldots;\boldsymbol{p}_j;\ldots;\boldsymbol{p}_i;\ldots;\boldsymbol{p}_n\rangle \tag{3.14}$$

i.e. the order of the creation doesn't matter, the final state is the same. This is characteristic of bosons, hence, the Klein Gordon field, give rise to bosons obeying Bose-Einstein statistics.*

We may also define the number operator, which essentially counts the number of particles, as

$$N = \int d^3k\; N(\boldsymbol{p}) = \int d^3k\; a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) \tag{3.15}$$

with

$$N\,|\boldsymbol{p}_1;\boldsymbol{p}_2;\ldots;\boldsymbol{p}_n\rangle = n\,|\boldsymbol{p}_1;\boldsymbol{p}_2;\ldots;\boldsymbol{p}_n\rangle \tag{3.16}$$

obeying the following commutation relations

$$[N, a^\dagger(\boldsymbol{p})] = a^\dagger(\boldsymbol{p}) \tag{3.17a}$$
$$[N, a(\boldsymbol{p})] = -a(\boldsymbol{p}) \tag{3.17b}$$

Finally, we can also give the expressions for the Hamiltonian eq. (3.4) and the 3-momentum eq. (3.5) in terms of $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$, which read

$$\mathrm{H} = \frac{1}{2}\int d^3p\; E_{\boldsymbol{p}}\left[a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) + a(\boldsymbol{p})a^\dagger(\boldsymbol{p})\right] \tag{3.18}$$

$$\boldsymbol{P} = \frac{1}{2}\int d^3p\; \boldsymbol{p}\left[a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) + a(\boldsymbol{p})a^\dagger(\boldsymbol{p})\right] \tag{3.19}$$

*In accordance to the fact that the KG equation is only valid to spin 0 particles.

3.2 Normal ordering

If one tries to evaluate the energy of the vacuum from eq. (3.18), the result is that it diverges due to the term $\langle 0|a(\boldsymbol{p})a^\dagger(\boldsymbol{p})|0\rangle = 1$. These are some of the reasons given in literature:

- *QFT Notes*, Eduard Massó [Mas19]: The divergence has two sources. The first is related to the fact that the harmonic oscillator has a non-vanishing zero-point energy, a field contains in all space-time an infinite amount of oscillators giving an infinite energy. The second source is due to the integration over all modes in the energy calculation, where the terms of higher frequency lead to infinite energy.
- *QFT Notes*, Oxford [Hai11]: This divergence arises because we want too much. We have assumed that our theory is valid to arbitrarily short distance scales, corresponding to arbitrarily high energies, which is clearly absurd. The integral should be cut off at high momentum, reflecting the fact that our theory presumably breaks down at some point (most likely far below the GUT or Planck scale).

To avoid this catastrophe, what we should do is renormalise the energy, set as the zero point energy the energy of the vacuum and measure the other energies with respect to this one, which is the only quantity that we can observe

$$E_q = \langle \boldsymbol{q}|\mathrm{H}|\boldsymbol{q}\rangle - \langle 0|\mathrm{H}|0\rangle \tag{3.20}$$

We introduce the *normal ordering* of operators, denoting the *normal product* as

$$\mathbf{N}\left[a(\boldsymbol{p}_1)a(\boldsymbol{p}_2)a^\dagger(\boldsymbol{p}_3)\right] = \mathbf{N}\left[a(\boldsymbol{p}_1)a(\boldsymbol{p}_2)a^\dagger(\boldsymbol{p}_3)\right] = a^\dagger(\boldsymbol{p}_3)a(\boldsymbol{p}_1)a(\boldsymbol{p}_2) \tag{3.21}$$

Practically, it consist on moving all the creation operators to the left and all annihilation operators to the right (but conserving the order between them).

From now on we will use this mathematical trick to calculate expectation values such like the energy of the vacuum state, which now is

$$E_0 = \langle 0|\mathbf{N}\left[\mathrm{H}\right]|0\rangle = 0 \tag{3.22}$$

In general, using normal ordering, we will have for a general state $|\boldsymbol{p}\rangle$

$$\mathrm{H}\,|\boldsymbol{p}\rangle = \mathbf{N}\left[\mathrm{H}\right]|\boldsymbol{p}\rangle = E_{\boldsymbol{p}}\,|\boldsymbol{p}\rangle \tag{3.23}$$

$$\boldsymbol{P}\,|\boldsymbol{p}\rangle = \mathbf{N}\left[\boldsymbol{P}\right]|\boldsymbol{p}\rangle = \boldsymbol{p}\,|\boldsymbol{p}\rangle \tag{3.24}$$

We may join the last two into one, using the covariant 4-momentum formulation where $P^\mu = (\mathrm{H}, \boldsymbol{P})$, we have

$$P^\mu = \mathbf{N}\left[P^\mu\right] = \int d^3p\; k^\mu N(\boldsymbol{p}) \tag{3.25}$$

that acts on a Fock state like

$$P^\mu\,|\boldsymbol{p}_1;\boldsymbol{p}_2;\ldots;\boldsymbol{p}_n\rangle = (p_1 + p_2 + \cdots + p_n)^\mu\,|\boldsymbol{p}_1;\boldsymbol{p}_2;\ldots;\boldsymbol{p}_n\rangle \tag{3.26}$$

3.3 Causality, time ordering and Feynamn propagator

To really discuss causality, we should ask not whether particles can propagate over space-like intervals, but whether a measurement performed at one point can affect a measurement at another point whose separation from the first is space-like. The simplest thing we could try to measure is the field $\phi(x)$, so we should compute the commutator $[\phi(x), \phi(y)]$ if this commutator vanishes, one measurement cannot affect the other. In fact, if the commutator vanishes for $(x-y)^2 < 0$ (space-like separation), causality is preserved quite generally, since commutators involving any function of $\phi(x)$, including $\pi(x) = \dot{\phi}(x)$, would also have to vanish. Of course we know from eq. (3.7) that the commutator vanishes for $x_0 = y_0$ (equal times); now let's do the more general computation:

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}\left[e^{-ip(x-y)} - e^{ip(x-y)}\right] = D(x-y) - D(y-x) \tag{3.27}$$

where we have defined the Lorentz invariant quantity

$$D(x-y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} e^{-ip(x-y)} \tag{3.28}$$

which expresses the amplitude for the particle to propagate from y to x.

Let's study the different cases separately:

a) Equal-time $x^0 = y^0$: when the two events happen at equal times, we expect that the commutator in eq. (3.27) vanishes, in accordance with eq. (3.7). We can factor out the dependency on the time component in this expression and obtain

$$[\phi(x), \phi(y)] \propto \int_{-\infty}^{\infty} \frac{d^3p}{\sqrt{\boldsymbol{p}^2 + m^2}} \sin(\boldsymbol{p} \cdot (\boldsymbol{x} - \boldsymbol{y})) = 0 \tag{3.29}$$

as the integrand is odd. So causality is preserved in this case.

b) Space-like $(x-y)^2 < 0$: a Lorentz transformation can be performed on the second term of eq. (3.27) changing $x - y \to y - x$, therefore $D(x-y) = D(y-x)$ and the commutator will be identically 0, preserving causality.

c) Time-like $(x-y)^2 > 0$: causality is preserved, just because the two points are causally connected and there exist NO Lorentz transformation that takes $x - y \to y - x$, so the commutator is allowed to have a nonzero value. In fact, the value of the commutator in this case give us the Feynman propagator Δ for mesons.

To summarise, the previous results can be written in compact form like

$$[\phi(x), \phi(y)] = \begin{cases} 0 & (x-y)^2 < 0 \\ i\delta^3(\boldsymbol{x} - \boldsymbol{y}) & x^0 = y^0 \\ i\Delta(x-y) & (x-y)^2 > 0 \end{cases} \tag{3.30}$$

The Feynman propagator $\Delta(x-y)$ is exactly the function that describes the propagation of a particle created at the point x and annihilated at y (if $x^0 < y^0$ and viceversa if $x^0 > y^0$). The expression for the propagation of a meson in the Klein-Gordon field is

$$i\Delta(x-y) \equiv \langle 0|\mathbf{T}\{\phi(x)\phi(y)\}|0\rangle = \theta(x^0 - y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0 - x^0)\langle 0|\phi(y)\phi(x)|0\rangle \tag{3.31}$$

where $\mathbf{T}\{-\}$ is the time ordering operator which time orders the operators with the latest happening to the left and $\theta(x)$ the Heaviside step function.

The explicit expression for the propagator follows from eqs. (3.28) and (3.31)

$$i\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left[\theta(x^0 - y^0)e^{-ip(x-y)} + \theta(y^0 - x^0)e^{ip(x-y)}\right] \tag{3.32}$$

with only one of the two terms surviving once x and y are fixed. It somehow represents the fluctuations of the vacuum as it expresses the propagation of *virtual particles* which are causally connected.

To obtain the relation between the Feynman propagator and the Klein-Gordon equation we should work out the expression

$$(\partial^2 + m^2)\Delta(x-y)$$

with the propagator given by the time ordering of the fields in eq. (3.31). It is convenient to introduce the identity

$$\partial_t \mathbf{T}\{A(t), B(t')\} = \mathbf{T}\left\{\dot{A}(t)B(t')\right\} + \delta(t-t')[A(t), B(t')] \tag{3.33}$$

where the derivative is respect to the time inside $\phi(x)$, this is why there is no time derivative of the second field which is evaluated at y. Also note that, the spatial derivative commute with the

time ordering, i.e. $\partial_i \mathbf{T}\{\phi(x)\phi(y)\} = \mathbf{T}\{[\partial_i\phi(x)]\phi(y)\}$, again $\partial_i = \partial/\partial x$. Thus, for our case, we have

$$\partial_0\Delta(x-y) = \mathbf{T}\left\{\dot{\phi}(x)\phi(y)\right\} + \delta(x^0-y^0)\cancelto{0}{[\phi(x),\phi(y)]}$$
$$\partial_0^2\Delta(x-y) = \mathbf{T}\left\{\ddot{\phi}(x)\phi(y)\right\} + \delta(x^0-y^0)[\dot{\phi}(x),\phi(y)] \overset{(3.7)}{=} \mathbf{T}\left\{\ddot{\phi}(x)\phi(y)\right\} - i\delta^4(x-y)$$

In the last equality we have used the canonical commutation relations and the fact that $\delta(x^0 - y^0)\delta^3(\boldsymbol{x}-\boldsymbol{y})$. Putting together all the results we have

$$(\partial_0^2 - \partial_i^2 - m^2)(-i)\mathbf{T}\{\phi(x)\phi(y)\} = -i\mathbf{T}\left\{[(\partial_0^2 - \partial_i^2 - m^2)\phi(x)]\phi(y)\right\} - \delta^4(x-y)$$

The first term cancels because $\phi(x)$ itself is a solution to the KG equation and we are left with the relation

$$(\partial^2 + m^2)\Delta(x-y) = -\delta^4(x-y) \tag{3.34}$$

so the propagator is the Green's function of the Klein-Gordon equation.* Nevertheless, it is convenient to change to the momentum (or Fourier) space as expressions adopt a much simpler form. By defining the propagator in momentum space by

$$\Delta(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx}\Delta(p) \tag{3.35}$$

the KG equation reads

$$(-p^2 + m^2)\Delta(p) = -1 \tag{3.36}$$

which is a simple algebraic equation whose solution is

$$\Delta(p) = \frac{1}{p^2 - m^2} \tag{3.37}$$

Obviously, this last expression is equivalent to eq. (3.31) in coordinate space, we just have to take the inverse Fourier transform of eq. (3.37) to recover the original expression†.

The propagator is extremely useful when we consider that there are sources of field $j(x)$ in the system, this is the case of the inhomogeneous KG equation

$$(\partial^2 + m^2)\phi(x) = j(x) \tag{3.38}$$

By the method of the Green function, which is the propagator, the equation can be solved and the result reads

$$\phi(x) = \phi_0(x) - \int d^4x'\Delta(x-x')j(x') \tag{3.39}$$

It is useful to visualise it in terms of Feynman diagrams. For $y^0 < x^0$, the vacuum expectation value becomes $\langle 0|\psi(x)\psi(y)|0\rangle$ and it is interpreted as the creation of a meson at y that propagates through space-time until it is annihilated at x. On the other hand, for $y^0 > x^0$, the surviving term in eq. (3.31) is $\langle 0|\psi(y)\psi(x)|0\rangle$, the propagation is pictured as a meson that propagates from x (where it is created) to y (where it is annihilated).

A comment on the origin of the propagator [HM84] The reader must be familiar with the perturbation theory of the Schrodinger equation. Consider the second order term of the expansion, its relativistic generalisation is

$$T_{fi}^{(2)} = -i\sum_{n\neq i} V_{fn}\frac{1}{E_i - E_n}V_{ni}2\pi\delta(E_f - E_i)$$

*This result can be generalised to any propagator in QFT.

†This is not so trivial as the integral has two divergences over the real axes which must be avoided by expanding the integration over the complex plane and making use of the residual theorem.

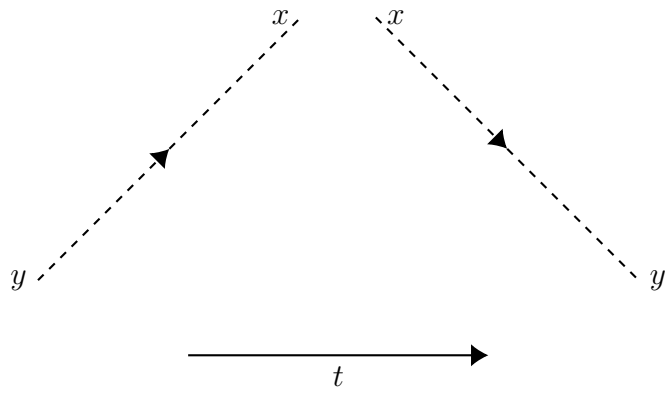


Figure 3.1: Left: $y^0 < x^0$, meson propagated from y to x. Right: $y^0 > x^0$, meson is propagated from x to y.

Apart from the δ, the rest is equal to the non-relativistic formula, we have just introduced the δ in the energies of the final and initial state just to impose energy conservation.*

Later on, we will see that a Feynman diagram is a sum over all possible time-ordered diagrams, i.e. of all the orders in the expansion. When we talk about the order, we are referring to the number of vertices and, equivalently, the number of propagators (of virtual particles to be transmitted). So, the second order term implies that there are 2 vertices so it can be written as

$$T_{fi}^{(2)} \sim V_{fc}\frac{1}{E_i - E_c}V_{ci} + V_{fc}\frac{1}{E_i - 2E_i - E_c}V_{ci} = V_{fc}\frac{2E_c}{E_i^2 - E_c^2}V_{ci}$$

where c is the virtual particle that propagates during the interaction. To determine the propagator, we calculate

$$E_i^2 = (p_A + p_B)^2 + (\boldsymbol{p}_A + \boldsymbol{p}_B)^2 = (p_A + p_B)^2 + \boldsymbol{p}^2$$
$$E_c^2 = m_c^2 + \boldsymbol{p}^2$$

so the term in the perturbation is

$$T_{fi}^{(2)} \sim V_{fc}\frac{2E_c^2}{p^2 - m_c^2}V_{ci}$$

where $p = p_A + p_B$ is the total initial 4-momentum. We see that inside this term there is hidden the propagator of eq. (3.37).

3.4 Complex field

Previously, in eq. (3.8) we imposed the field to be real, we will remove this constraint and consider fields where $\phi \neq \phi^\dagger$. The Lagrangian density function for a complex KG field is

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi \tag{3.40}$$

Note that the complex field can always be expanded into two real fields $\phi(x) = (\phi_R(x) + i\phi_I(x))/\sqrt{2}$ giving rise to two separate Lagrangians $\mathcal{L} = \mathcal{L}_R(\phi_R) + \mathcal{L}_I(\phi_I)$ which proves that the whole of it is real.

The equation of motion for the complex field is equal to that of the real field (eq. (3.2))

$$(\partial^2 + m^2)\phi = 0 \tag{3.41}$$

Also, the conjugate momenta is $\pi = \dot{\phi}^\dagger$ and the Hamiltonian of eq. (3.4) is valid. However, the Fourier expansion changes

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_{\boldsymbol{p}}}}\left[a(\boldsymbol{p})e^{-ipx} + b^\dagger(\boldsymbol{p})e^{ipx}\right] \tag{3.42}$$

*This process is known as old-fashioned quantum field theory (OQFT).

where two creation operators $a^\dagger(\boldsymbol{p}), b^\dagger(\boldsymbol{p})$ and two annihilation operators $a(\boldsymbol{p}), b(\boldsymbol{p})$ exist.

By imposing the same equal time commutation relations eq. (3.7), $[\phi(x), \phi^\dagger(y)]$, we obtain the commutation relations that verify the creation and annihilation operators:

$$\begin{aligned}
&[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = \delta^3(\boldsymbol{p} - \boldsymbol{p}') \\
&[b(\boldsymbol{p}), b^\dagger(\boldsymbol{p}')] = \delta^3(\boldsymbol{p} - \boldsymbol{p}') \\
&[a(\boldsymbol{p}), b(\boldsymbol{p}')] = [a^\dagger(\boldsymbol{p}), b^\dagger(\boldsymbol{p}')] = 0
\end{aligned} \tag{3.43}$$

and any other relation not listed is identical to 0.

We return to the problem of causality, from eq. (3.42) we see that the action of the field $\phi(x)$ is to create particles of time b and destroy particles of type a at position x, while $\phi^\dagger(y)$ creates particles of type a and destroy type b particles at position y. Then, the commutator $[\phi(x), \phi^\dagger(y)]$ will have nonzero contributions which must cancel outside the light cone. The first term represents the propagation of a negatively charged particle from y to x, the second term represents the propagation of a positively charged particle from x to y. In order for these two processes to be present and give cancelling amplitudes, both of these particles must exists, and they must have the same mass. In QFT, causality requires that every particle have a corresponding antiparticle with the same mass and opposite quantum numbers (for the real KG field, the particle is its own antiparticle) [PS95].

The vacuum state $|0\rangle$ is defined as that

$$a(\boldsymbol{p})\,|0\rangle = 0 \qquad \& \qquad b(\boldsymbol{p})\,|0\rangle = 0 \qquad \forall \boldsymbol{p} \tag{3.44}$$

The action of the creation operators is to create particles but now the particles created are different since we are using distinct creation operators, we will represent the two particles like

$$a^\dagger(\boldsymbol{p})\,|0\rangle = \left|\boldsymbol{p}, \pi^+\right\rangle \qquad , \qquad b^\dagger(\boldsymbol{p})\,|0\rangle = \left|\boldsymbol{p}, \pi^-\right\rangle$$

It follows that, by continuously applying the creation operators, the resulting state will be like

$$\left|\boldsymbol{p}_1, \pi^+; \boldsymbol{p}_2, \pi^+; \boldsymbol{p}_3, \pi^-; \boldsymbol{p}_4, \pi^+\right\rangle = a^\dagger(\boldsymbol{p}_1)a^\dagger(\boldsymbol{p}_2)b^\dagger(\boldsymbol{p}_3)a^\dagger(\boldsymbol{p}_4)\,|0\rangle \tag{3.45}$$

Finally, let us introduce the expression for the operators N, P^μ*

$$N = \int d^3p\; N(\boldsymbol{p}) = \int d^3p\; [N^+(\boldsymbol{p}) + N^-(\boldsymbol{p})] = \int d^3p\; [a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) + b^\dagger(\boldsymbol{p})b(\boldsymbol{p})] \tag{3.46}$$

$$P^\mu = \int d^3p\; p^\mu [N^+(\boldsymbol{p}) + N^-(\boldsymbol{p})] \tag{3.47}$$

which act on a state of the Fock space like in eq. (3.45) as

$$N\left|\boldsymbol{p}_1, \pi^\pm; \cdots; \boldsymbol{p}_n, \pi^\pm\right\rangle = n\left|\boldsymbol{p}_1, \pi^\pm; \cdots; \boldsymbol{p}_n, \pi^\pm\right\rangle \tag{3.48}$$

$$P^\mu\left|\boldsymbol{p}_1, \pi^\pm; \cdots; \boldsymbol{p}_n, \pi^\pm\right\rangle = \left(\sum_{j=1}^{n} p_j^\mu\right)\left|\boldsymbol{p}_1, \pi^\pm; \cdots; \boldsymbol{p}_n, \pi^\pm\right\rangle \tag{3.49}$$

Notice that it counts both, the number of particles of type π^+ and of type π^-, without any distinction. In particular, for H $= P^0$, the previous expression gives the same energy for both particles, $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$.

The question is which operator enables us to distinguish between particles? The answer arises naturally once we notice that the field has an internal symmetry of the type $\phi(x) \to e^{-iq\theta}\phi(x)$ that preserves the Lagrangian. Consequently, from Noether's theorem, the symmetry has associated a conserved current

$$j^\mu = -iq\left[(\partial^\mu\phi^\dagger)\phi - \phi^\dagger(\partial^\mu\phi)\right] \tag{3.50}$$

With this, we can compute the conserved charge Q that reads in momentum space

$$Q = q\int d^3p\; [N^+(\boldsymbol{p}) - N^-(\boldsymbol{p})] \tag{3.51}$$

*In both we have already applied normal ordering.

This equation means that, every time we add a particle of type π^+ the total charge increments by 1 unit of q but if we add a particle of type π^- the total charge reduces by 1 unit of q. Mathematically, this operator acts on the Fock states as

$$Q\left|\boldsymbol{p},\pi^{\pm}\right\rangle = \pm q\left|\boldsymbol{p},\pi^{\pm}\right\rangle \tag{3.52}$$

For the first time, we are able to distinguish between the two types of particles. Moreover, we have found an incredible result, if we associate Q with the total electric charge and q the elementary charge (which we can set $q = 1$), then the relationship between π^+ and π^- is that one is the antiparticle of the other because they both have the same mass and energy but opposite charge. It is a convention to choose π^- as the particle* and π^+ as the antiparticle. We write $\pi^- = \overline{\pi^+}$ and $\pi^+ = \overline{\pi^-}$ to express the antiparticle.

*This choice must be familiar to the reader as electrons have negative charge although this theory is only valid for spin 0 particles.

4 Dirac Field

In deriving the Dirac field, he sought to find an equation which was first order in both time and spatial coordinates. He took as a starting point the Klein-Gordon eq. (3.2), which can be expressed as

$$(i\beta^\mu\partial_\mu + m)(i\gamma^\nu\partial_\nu - m)\psi(x) = 0 \tag{4.1}$$

where β^μ and γ^ν are 4-vectors.

Notice that we have obtained two terms linear in ∂_μ so if we imposes that the field is a solution of the first term then the whole equation automatically vanishes and we are lead to the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \tag{4.2}$$

The next step is to find which conditions must verify the 4-vectors β^μ and γ^μ. We will do so by imposing that eq. (4.1) reduces to the KG equation (eq. (3.2)), after multiplying the terms we obtain

$$\left[\beta^\mu\gamma^\nu\partial_\mu\partial_\nu - i\left(\gamma^\mu - \beta^\mu\right)m + m^2\right]\phi(x) = 0 \tag{4.3}$$

This implies that $\beta^\mu = \gamma^\mu$ for the cross terms to vanish. Then, in the first term, all mixed terms $\mu \neq \nu$ must also vanish if we compare it with eq. (3.2) so $\gamma^\mu\gamma^\nu = 0$ and, for the case when $\mu = \nu$, the result of the multiplication must be proportional to the Minkowski metric $\eta^{\mu\nu}$. These two conditions can be reduced into 1, which is usually quoted as the fundamental property of the γ-matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{4.4}$$

where $\{A, B\} = AB + BA$ denoted the anticommutator. At this point, Dirac noted that the components of the 4-vector γ^μ could not be scalars and neither 2×2 matrices (for example the Pauli matrices with the identity, $\gamma^\mu = (\mathbb{I}, \boldsymbol{\sigma})$) as they do not verify eq. (4.4). He then though: "I suddenly realised that there was no need to stick to quantities which can be represented by matrices with just two rows and columns. Why not go to four rows and columns?". Indeed, this idea solved the problem and he found the explicit form of the γ matrices

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & \mathbb{0} \\ \mathbb{0} & \mathbb{I} \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} \mathbb{0} & \sigma^i \\ -\sigma^i & \mathbb{0} \end{pmatrix} \tag{4.5}$$

where all of the components inside are 2×2 matrices. It is customary to work using this notation as every matrix can be divided in 4 blocks and treat each of them independently of the other.

In fact, there are infinite choices of the γ matrices that differ from an unitary matrix of those in eq. (4.5). That choice is called Dirac representation which is more useful in the non-relativistic limit, there is also the Weyl (or chiral) representation, useful in the extreme relativistic case, where the γ matrices are

$$\gamma^0 = \begin{pmatrix} \mathbb{0} & \mathbb{I} \\ \mathbb{I} & \mathbb{0} \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} \mathbb{0} & \sigma^i \\ -\sigma^i & \mathbb{0} \end{pmatrix} \tag{4.6}$$

Together with the 4-matrices, we also define γ^5 as the complete antisymmetric combination of the 4 other matrices

$$\gamma^5 = \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho \tag{4.7}$$

and the set of matrices $\{\sigma^{\mu\nu}\}$ (do not confuse with the Pauli matrices σ^i) defined by

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{4.8}$$

Note that this is not a tensor as it doesn't transform like it should but we preserve the notation to lower the indices $\sigma_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}\sigma^{\alpha\beta}$.

This matrices are in fact the boost and rotation generator of the Lorentz group. A 4-vector field $\phi(x)$ that transforms under boosts and rotations according to $\sigma^{\mu\nu}$ is called a *Dirac spinor*. Note that, the rotation generators $\sigma^{\mu\nu}$ is just a 3-dimensional spinor transformation matrix replicated twice (remember that Pauli matrices where the generators of rotations in $SU(2)$).

We also introduce the matrix vector $\mathbf{\Sigma}$ as a generalisation to 4 dimensions of the spin operator $\boldsymbol{\sigma}$ for two spinors. Its components are given by

$$\Sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma^{jk} \tag{4.9}$$

which in our case gives the 3 component vector

$$\mathbf{\Sigma} = (\sigma^{23}, \sigma^{31}, \sigma^{12}) \tag{4.10}$$

Comment on Dirac's equation In eq. (4.2) we have defined Dirac equation, without knowledge of what the hell where those γ's but know we know that are matrices so, how can we subtract a scalar (the mass) to a matrix? To fix this, we should implicitly read Dirac equation as $(i\partial_\mu - m\mathbb{I}_n)\phi(x) = \mathbb{0}_n$ where $\mathbb{I}_n$ is the identity matrix of the space where the γ matrices live, essentially $\mathbb{C}^{n\times n}$, the same with $\mathbb{0}_n$ and $\phi(x)$ an n-dimensional complex field. Similarly, the right hand side of eq. (4.4) there appears a number $\eta^{\mu\nu}$ while the left hand side is a 4×4 matrix, therefore the right hand side must be read as $\eta^{\mu\nu}\mathbb{I}_n$ although we don't write the identity matrix.

More over, as the contraction with the γ matrices occurs quite often, Feynman invented the slash notation

$$\not{A} = \gamma^\mu A_\mu \tag{4.11}$$

So Dirac's equation is completely simplified by writing $\not{\partial} = \gamma^\mu\partial_\mu$ giving

$$(i\not{\partial} - m)\psi(x) = 0 \tag{4.12}$$

The solution $\psi(x)$ is called a Dirac spinor, and from the block construction of the γ matrices, it can be represented as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{4.13}$$

where ψ_L and ψ_R are two 2-dimensional Pauli spinors. Very roughly speaking, the four components of the bispinor account for the two possible spin states of a fermion, along with the two possibilities of a particle or an antiparticle. We have also added a bit of suggestive notation in terms of subscripts, since the L and R refer to the handedness of a particle. Though we haven't yet derived the properties of spin, we will find that a right-handed particle's intrinsic angular momentum is in the same direction as the spin (as given by the right-hand rule), and that of a left-handed particle is the the reverse. [Gol17]

And finally, what type of particles does Dirac equation describe? For the case $n = 4$, which is the case we will study, Dirac equation serve to explain spin 1/2 particles. But not only this, because only 2 degrees of freedom (DoF) are needed to describe those types of particles, the fact that we have 4 DoF implies that we give rise to a second particle (with the same mass, energy...) which is the antiparticle of the first one.

Properties of γ matrices The basic properties of the γ matrices to keep in mind are:

Square	$(\gamma^\mu)^2 = \eta^{\mu\mu}\mathbb{I}_4$
Anticommute	$\{\gamma^\mu, \gamma^\nu\} = 0\ \mu \neq \nu \Rightarrow \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$
Trace	$\mathrm{Tr}\,\gamma^\mu = 0$
Dagga	$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$
γ^5	$(\gamma^5)^2 = 1,\ \gamma^5 = (\gamma^5)^\dagger$ & $\{\gamma^5, \gamma^\mu\} = 0$
Slash product	$\not{a}\not{b} = ab - i\sigma^{\mu\nu}a_\mu b_\nu$
Commutator	$[\gamma^5, \sigma^{\mu\nu}] = 0$
Product $\sigma^{\mu\nu}\gamma_5$	$\sigma^{\mu\nu}\gamma_5 = (-i/2)\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}$

For completeness, let us recap here the complete form of all the matrices defined in the Dirac and Weyl representation:

	Dirac	Weyl
γ^0	$\begin{pmatrix} \mathbb{I} & \mathbb{0} \\ \mathbb{0} & \mathbb{I} \end{pmatrix}$	$\begin{pmatrix} \mathbb{0} & \mathbb{I} \\ \mathbb{I} & \mathbb{0} \end{pmatrix}$
γ^i	$\begin{pmatrix} \mathbb{0} & \sigma^i \\ -\sigma^i & \mathbb{0} \end{pmatrix}$	$\begin{pmatrix} \mathbb{0} & \sigma^i \\ -\sigma^i & \mathbb{0} \end{pmatrix}$
γ^5	$\begin{pmatrix} \mathbb{0} & \mathbb{I} \\ \mathbb{I} & \mathbb{0} \end{pmatrix}$	$\begin{pmatrix} -\mathbb{I} & \mathbb{0} \\ \mathbb{0} & \mathbb{I} \end{pmatrix}$
σ^{0i}	$i\begin{pmatrix} \mathbb{0} & \sigma^i \\ \sigma^i & \mathbb{0} \end{pmatrix}$	$i\begin{pmatrix} -\sigma^i & \mathbb{0} \\ \mathbb{0} & \sigma^i \end{pmatrix}$
σ^{ij}	$\epsilon^{ijk}\begin{pmatrix} \sigma^k & \mathbb{0} \\ \mathbb{0} & \sigma^k \end{pmatrix}$	$\epsilon^{ijk}\begin{pmatrix} \sigma^k & \mathbb{0} \\ \mathbb{0} & \sigma^k \end{pmatrix}$

Lagrangian The lagrangian density for the Dirac equation is

$$\mathcal{L} = \frac{i}{2}\overline{\psi}\overleftrightarrow{\partial\!\!\!/}\psi - m\overline{\psi}\psi \tag{4.14}$$

where we have defined the adjoint field $\overline{\psi}$ as

$$\overline{\psi} = \psi^\dagger\gamma^0 \tag{4.15}$$

and the meaning of the $\leftrightarrow$ over the derivative is used to indicate that it is acting on both sides of the equation, i.e.

$$\overline{\psi}\overleftrightarrow{\partial\!\!\!/}\psi = \overline{\psi}(\partial\!\!\!/\psi) - (\partial\!\!\!/\overline{\psi})\psi \tag{4.16}$$

The reason for using the adjoint, instead of simply the $\psi^\dagger$ in eq. (4.18), is because the product $\psi^\dagger\psi$ is not a Lorentz while $\overline{\psi}\psi$ is.

The Dirac equation for the field (eq. (4.2)) by applying the Euler-Lagrange equation eq. (1.4) to the adjoint field $\overline{\psi}$. However, we could have also taken the Euler-Lagrange equations from the field itself, obtaining the adjoint Dirac equation

$$\overline{\psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0 \tag{4.17}$$

where here $\overleftarrow{\partial\!\!\!/}$ denotes that the derivative acts on the left.

Although, eq. (4.18) is hermitian, usually it is used for practical calculations the more user friendly, but not hermitian, version

$$\mathcal{L}_1 = \overline{\psi}(x)(i\partial\!\!\!/ - m)\psi(x) \tag{4.18}$$

Hamiltonian From the Lagrangian density, it is not hard to find the Hamiltonian for the system. First of all, find using eq. (4.18) that the conjugate momentum to ψ_a is

$$\pi_a = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_a} = i\psi_a^\dagger \tag{4.19}$$

where the label a runs from 1 to 4 and indicates the position in the Dirac spinor. Then, separating the $\partial\!\!\!/$ into time and space and substituting by the conjugate momentum we arrive to the Hamiltonian

$$H = i\int d^3x\psi^\dagger(x)\frac{\partial}{\partial t}\psi(x) \tag{4.20}$$

By covariazing the later result we deduce that the 4-momentum operator for the Dirac field is

$$P^\mu = i\int d^3x\psi^\dagger(x)\partial^\mu\psi(x) \tag{4.21}$$

4.1 Solutions to Dirac equation

4.1.1 Stationary particles

The solutions to the Dirac equation are represented by a complete set of orthonormal states $\{u_\pm(\boldsymbol{p}), v_\pm(\boldsymbol{p})\}$ that span the 4-dimensional space. Consider a bispinor field that's entirely independent of position, in which case the Dirac equation reduces to

$$(i\gamma^0\partial_0 - m)\psi = 0 \longrightarrow \begin{cases} i\dot{\psi}_L - m\psi_R &= 0 \\ i\dot{\psi}_R - m\psi_L &= 0 \end{cases} \tag{4.22}$$

which quickly yield four linearly independent solutions,

$$u_\pm(x) = \sqrt{m}\begin{pmatrix}\xi_\pm \\ \xi_\pm\end{pmatrix}e^{-imt} \quad , \quad v_\mp(x) = \sqrt{m}\begin{pmatrix}\xi_\pm \\ -\xi_\pm\end{pmatrix}e^{imt} \tag{4.23}$$

where $\xi_\pm$ are the eigenstates of the 3rd Pauli matrix

$$\xi_+ = \begin{pmatrix}1\\0\end{pmatrix} \quad , \quad \xi_- = \begin{pmatrix}0\\1\end{pmatrix} \tag{4.24}$$

where $\sigma_z\xi_\pm = \pm\xi_\pm$. The two-fold degeneracies in the energy for a given $\boldsymbol{p}$ results from the two possible spin orientations.

Applying the Hamiltonian, i.e. P^0, reveals the energy of the particles

$$\mathrm{H}u_\pm = mu_\pm \tag{4.25a}$$
$$\mathrm{H}v_\pm = -mv_\pm \tag{4.25b}$$

For the v states, the energy of a stationary particle (the mass term) is negative. We saw earlier that these are the positrons, and they present their own particular problems.

4.1.2 Free particles

Assuming that the solutions must be of the form of plane waves, $\psi(p) \propto e^{\pm ipx}$, yields (in the Weyl representation*) the coupled set of equations

$$\begin{pmatrix} -m & \pm(p_0 - \boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ \pm(p_0 + \boldsymbol{\sigma}\cdot\boldsymbol{p}) & -m \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = 0 \tag{4.26}$$

From the first and second line, independently, we obtain the relations

$$\psi_L = \mp\frac{p_0 - \boldsymbol{\sigma}\cdot\boldsymbol{p}}{m}\psi_R \quad , \quad \psi_R = \mp\frac{p_0 + \boldsymbol{\sigma}\cdot\boldsymbol{p}}{m}\psi_L \tag{4.27}$$

where the $+$ sign corresponds to positive energy solutions ($u_\pm e^{-ipx}$) and the $-$ to negative energy ($v_\pm e^{ipx}$). In the notation of a Dirac spinor this 4 solutions read

$$u_\pm(p) = \frac{m}{\sqrt{E_{\boldsymbol{p}} \pm |\boldsymbol{p}|}}\begin{pmatrix}\xi_\pm \\ \frac{E_{\boldsymbol{p}}\pm\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m}\xi_\pm\end{pmatrix} \quad , \quad v_\mp(p) = \frac{m}{\sqrt{E_{\boldsymbol{p}} \pm |\boldsymbol{p}|}}\begin{pmatrix}\xi_\pm \\ -\frac{E_{\boldsymbol{p}}\pm\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m}\xi_\pm\end{pmatrix} \tag{4.28}$$

The normalisation of the spinors in eq. (4.23) and eq. (4.28) is chosen according to the following relations

$$\left.\begin{aligned} u_r^\dagger(\boldsymbol{p})u_s(\boldsymbol{p}) = v_r^\dagger(\boldsymbol{p})v_r(\boldsymbol{p}) = 2E_{\boldsymbol{p}}\delta_{rs} \\ u_r^\dagger(\boldsymbol{p})v_s(-\boldsymbol{p}) = 0 \end{aligned}\right\} \longleftrightarrow \begin{cases} \overline{u}_r(\boldsymbol{p})u_s(\boldsymbol{p}) = -\overline{v}_r(\boldsymbol{p})v_r(\boldsymbol{p}) = 2m\delta_{rs} \\ \overline{u}_r(\boldsymbol{p})v_s(\boldsymbol{p}) = 0 \end{cases} \tag{4.29}$$

and satisfy the completeness relation

$$\sum_{r=\pm}\left[u_{r,a}(\boldsymbol{p})\overline{u}_{r,b}(\boldsymbol{p}) - v_{r,a}(\boldsymbol{p})\overline{v}_{r,b}(\boldsymbol{p})\right] = \delta_{ab} \tag{4.30}$$

*See Section 3.14 in [Mas19] for the solutions in the Dirac representation.

Helicity We can combine the spin operator with the momentum of a particle to produce a helicity operator:

$$\hat{h} = \frac{\boldsymbol{\Sigma} \cdot \boldsymbol{p}}{|\boldsymbol{p}|} = \boldsymbol{\Sigma}_{\boldsymbol{p}} \tag{4.31}$$

The helicity operator produces $+1/2$ for a right-handed beam and $-1/2$ for a left-handed one. This is the reason for choosing such labels for the Dirac spinor eq. (4.13).

Relativistic (and only relativistic) fermions are helicity eigenstates. If the spin and momentum are aligned, the particle is considered right-handed in the sense that a particle propagating in the direction given by your right-hand thumb will be spinning in the direction indicated by the curl of your fingers. The left handed case (u_- propagating in the +z-direction, for instance) is the same, but for your left hand. These results summarise as

$$\boldsymbol{\Sigma}_{\boldsymbol{p}} u_\pm = \pm \frac{1}{2} u_\pm \tag{4.32a}$$

$$\boldsymbol{\Sigma}_{\boldsymbol{p}} v_\pm = \mp \frac{1}{2} v_\pm \tag{4.32b}$$

Helicity is a useful property of particle because it is a conserved quantity. However, it is not conserved under Lorentz transformation. Consider a particle moving in the $+z - direction$ with spin $+1/2$, if we measure the particle in its rest frame we will find this exact value for the spin. Now, change to a moving frame were we see the particle moving in the $-z$-direction, the value of the spin remains $1/2$ but the other part has changed sign. Therefore, we conclude that the value of helicity is can change depending on the reference frame. Only in cases where the particle travels at the speed of light (massless particles), so there is no reference frame where the particle has flipped the direction of motion, helicity will be conserved in time and under Lorentz boosts.

4.2 Quantization

Like we did for the complex Klein-Gordon field, we expand the Dirac field in momentum space using the Fourier transform

$$\psi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_{\boldsymbol{p}}}} \sum_{r=\pm} \left[c_r(\boldsymbol{p}) u_r(\boldsymbol{p}) e^{-ipx} + d_r^\dagger(\boldsymbol{p}) v_r(\boldsymbol{p}) e^{ipx} \right] = \psi^+(x) + \psi^-(x) \tag{4.33}$$

where ψ^+ (ψ^-) contains the positive (negative) energy part, and therefore, all the annihilation (creation) operators $c_\pm(\boldsymbol{p})$ ($d_\pm^\dagger(\boldsymbol{p})$). The same for the conjugate field

$$\overline{\psi}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_{\boldsymbol{p}}}} \sum_{r=\pm} \left[c_r^\dagger(\boldsymbol{p}) \overline{u}_r(\boldsymbol{p}) e^{ipx} + d_r(\boldsymbol{p}) \overline{v}_r(\boldsymbol{p}) e^{-ipx} \right] = \overline{\psi}^-(x) + \overline{\psi}^+(x) \tag{4.34}$$

To quantize the field, similar to what we did for the KG field in eq. (3.7), we impose the equal-time canonical anticommutation relations

$$\begin{aligned} \{\psi_a(t, \boldsymbol{x}), \psi_b^\dagger(t, \boldsymbol{y})\} &= \delta_{ab} \delta^3(\boldsymbol{x} - \boldsymbol{y}) \\ \{\psi_a(t, \boldsymbol{x}), \psi_b(t, \boldsymbol{y})\} &= \{\psi_a^\dagger(t, \boldsymbol{x}), \psi_b^\dagger(t, \boldsymbol{y})\} = 0 \end{aligned} \tag{4.35}$$

which imply the following anticommutation relations in momentum space

$$\begin{aligned} \{c_r(\boldsymbol{p}), c_s^\dagger(\boldsymbol{p}')\} &= \delta_{rs} \delta^3(\boldsymbol{p} - \boldsymbol{p}') \\ \{d_r(\boldsymbol{p}), d_s^\dagger(\boldsymbol{p}')\} &= \delta_{rs} \delta^3(\boldsymbol{p} - \boldsymbol{p}') \end{aligned} \tag{4.36}$$

with the other combinations vanishing.

Fock space This operators are defined like all creation and annihilation operators that the reader must have seen in his life. In particular, the vacuum state is such that

$$c_r(\boldsymbol{p}) |0\rangle = d_r(\boldsymbol{p}) |0\rangle = 0 \qquad \forall \boldsymbol{p} \tag{4.37}$$

or in position space

$$\psi^+(x)\,|0\rangle = \overline{\psi}^+(x)\,|0\rangle = 0 \qquad \forall x \tag{4.38}$$

A state in this space is a concatenation of applications of $c_r^\dagger$ and $d_s^\dagger$

$$|\boldsymbol{p}_1, r_1; \boldsymbol{p}_2, r_2; \cdots; \boldsymbol{p}_n, r_n\rangle = c_{r_1}^\dagger(\boldsymbol{p}_1) d_{r_2}^\dagger(\boldsymbol{p}_2) \cdots c_{r_n}^\dagger(\boldsymbol{p}_n) \tag{4.39}$$

Let's have a look at what happens when we exchange two particles $i \leftrightarrow j$ or the same type. In a system with only two particles

$$|\boldsymbol{p}, r; \boldsymbol{p}', s\rangle = c_r^\dagger(\boldsymbol{p}) c_s^\dagger(\boldsymbol{p}')\,|0\rangle \overset{(4.36)}{=} -c_s^\dagger(\boldsymbol{p}') c_r^\dagger(\boldsymbol{p})\,|0\rangle = |\boldsymbol{p}, r; \boldsymbol{p}, s\rangle - |\boldsymbol{p}', s; \boldsymbol{p}, r\rangle \tag{4.40}$$

There is a minus sign in discordance, the state is antisymmetric under the interchange of particle labels as required for fermions. If $\boldsymbol{p} = \boldsymbol{p}'$ and $r = 0$, the last results implies that

$$|\boldsymbol{p}, r; \boldsymbol{p}; r\rangle = 0 \tag{4.41}$$

Thus regaining Pauli's exclusion principle saying that two particles cannot exist in the same single particle state.

All of this is a consequence of the quantization where we used anticommutators instead of commutators. In a sentence, commutators lead to Bose-Einstein statistics while anticommutators to Fermi-Dirac statistics. However, this implication can not be deduce from a mathematical perspective, it is just an association that we can make given what physicist observe in nature.

Define the number operators

$$N_r(\boldsymbol{p}) = c_r^\dagger(\boldsymbol{p}) c_r(\boldsymbol{p}) \qquad , \qquad \overline{N}_r(\boldsymbol{p}) = d_r^\dagger(\boldsymbol{p}) d_r(\boldsymbol{p}) \tag{4.42}$$

These two creation and annihilation operators create two different types of particles. However, they both share some common characteristics. For example, they have the same mass as both fields are solutions from the same Dirac equation (eq. (4.2)) and their spin is the same. But, as noted in eq. (4.25), one has positive energy while the other has negative energy. This is a problem as the energy could go to $-\infty$, thus any state would be stable. How to fix this? Well, normal order...

Using the expansion in eq. (4.33), we obtain an expression for the 4-momentum in terms of the creation/annihilation operators

$$P^\mu = \int d^3p\; p^\mu \sum_{r=\pm} \left[c_r^\dagger(\boldsymbol{p}) c_r(\boldsymbol{p}) - d_r(\boldsymbol{p}) d_r^\dagger(\boldsymbol{p})\right] \tag{4.43}$$

When taking the expectation value of this with the vacuum state, the term in c's cancels but the second term diverges. Thus, we must introduce the normal ordering to this expression to find the correct value. Its definition is a bit different than the one given for the KG field as the operators $c_r(\boldsymbol{p})$ and $d_r(\boldsymbol{p})$ anticommute thus, whenever we exchange two fields or operators, we must introduce a minus sign in front of it:

$$\mathbf{N}\left[\psi_a \psi_b\right] = \mathbf{N}\left[\psi_a^+\psi_b^+ + \psi_a^-\psi_b^+ + \psi_a^+\psi_b^- + \psi_a^-\psi_b^-\right] = \psi_a^+\psi_b^+ + \psi_a^-\psi_b^+ - \psi_b^-\psi_a^+ + \psi_a^-\psi_b^- \tag{4.44}$$

In the case of matter, the term $d_r^\dagger(\boldsymbol{p}) d_r(\boldsymbol{p})$, once normal ordered, will turn to $-d_r^\dagger(\boldsymbol{p}) d_r(\boldsymbol{p}) = -\overline{N}_r$ so the 4-momentum normal ordered is

$$\mathbf{N}\left[P^\mu\right] = \int d^3p\; p^\mu \sum_r \left[N_r(\boldsymbol{p}) + \overline{N}_r(\boldsymbol{p})\right] \tag{4.45}$$

In particular, the energy of the whole system is

$$\mathrm{H} = \mathbf{N}\left[P^0\right] = \int d^3p\; E_{\boldsymbol{p}} \sum_r \left[N_r(\boldsymbol{p}) + \overline{N}_r(\boldsymbol{p})\right] \tag{4.46}$$

Note that the two types of particles contribute in the same manner to the final energy of the system, in other words, there is no way to distinguish the two particles by just measuring the

energy of the system. This is clearly seen in the definition of the Fock state eq. (4.39), while on the right we have applied the two creation operators, this is not reflected on the left as both are represented in the same way by their momentum and spin.

We must find an operator that can distinguish between the two particles. As in the case for the complex KG field, this operator is the electric charge Q, defined in the position and momentum space as

$$Q = q \int d^3x \; \psi^\dagger \psi = q \int d^3p \; \sum_{r=\pm} \left[N_r(\boldsymbol{p}) - \overline{N}_r(\boldsymbol{p}) \right] \tag{4.47}$$

The difference is clearly seen, if we add a particle of type c the charge $Q = q$ but if we add a particle of type d then $Q = -q$. If we identity the mass in Dirac's equation (4.2) as the electron mass, then $q = -e$ and the two particles can be identified as electrons and positrons. With this result, a particle state can no longer be specified with the momentum and spin only but we must also use the electric charge

$$|\boldsymbol{p}, r, q\rangle = c_r^\dagger(\boldsymbol{p}) \, |0\rangle \qquad , \qquad |\boldsymbol{p}, r, -q\rangle = d_r^\dagger(\boldsymbol{p}) \, |0\rangle \tag{4.48}$$

4.3 The fermionic propagator

The Dirac propagator, valid for spin 1/2 particles, is defined in a similar way as the KG propagator eq. (3.31),

$$iS_F(x-y) = \langle 0 | \mathbf{T} \left\{ \psi(x) \overline{\psi}(y) \right\} | 0 \rangle \tag{4.49}$$

where the time ordering for Dirac fields is

$$\mathbf{T} \left\{ \psi(x) \overline{\psi}(x) \right\} = \theta(x^0 - y^0) \psi(x) \overline{\psi}(y) - \theta(y^0 - x^0) \overline{\psi}(x) \psi(x) \tag{4.50}$$

This change in sign, in front of the term $y^0 > x^0$, is a consequence of the anticommutation relations imposed for the Dirac fields eq. (4.35).

An expression for the propagator is obtained by the method of the Greens function for eq. (4.2)

$$(i\partial\!\!\!/ - m) S_F(x-y) = \delta(x-y) \tag{4.51}$$

This partial differential equation is easily solved in momentum space, with the Fourier transform of $S_F(x)$

$$S_F(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} S_F(p) \tag{4.52}$$

which gives, once substituted in eq. (4.51)

$$iS_F(p) = \frac{i}{p\!\!\!/ - m + i\epsilon} = i \frac{p\!\!\!/ + m}{p^2 - m^2 + i\epsilon} \tag{4.53}$$

As for the meson propagator, it is useful to visualise the fermion propagator in terms of Feynman diagrams. For $y^0 < x^0$, only the first term in eq. (4.50) survive and it is interpreted as the creation of a e^- in x that propagates through space-time until it is annihilated in y. On the other hand, for $y^0 > x^0$, the propagation is pictured as a positron that propagates backwards in time from x (where it is created) to y (where it is annihilated).

Note that in the diagrams, the fermion line points from the vertex associated with the $\overline{\psi}(y)$ field to the vertex associated with the $\psi(x)$ field, i.e. the arrow runs in the same direction as time for electrons but in the opposite direction for positrons.

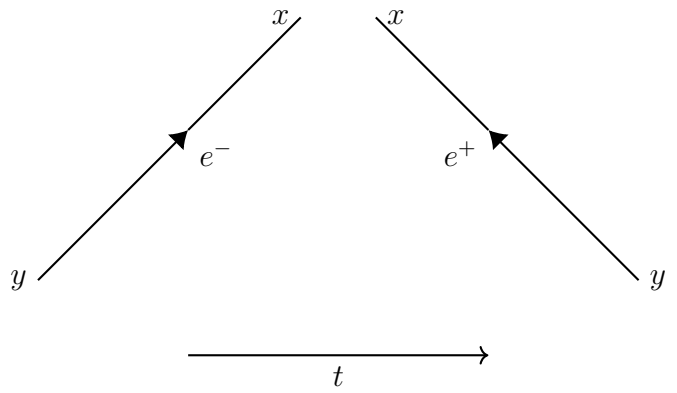


Figure 4.1: Left: $y^0 < x^0$, electron is propagated from y to x. Right: $y^0 > x^0$, positron is propagated from x to y.

5 The EM field

We shall develop a covariant theory of electromagnetism starting from the classical point of view, starting with the explicit fields $\boldsymbol{E}$ and $\boldsymbol{B}$ to the relativistic formulation using the 4-potential $A^\mu(x) = (\phi, \boldsymbol{A})$.

5.1 The classical field

The Maxwell's equations for the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields read

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho(t, \boldsymbol{x}) \tag{5.1a}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{5.1b}$$
$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{5.1c}$$
$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{j}(t, \boldsymbol{x}) \tag{5.1d}$$

where $\rho(t, \boldsymbol{x})$ is the charge density and $\boldsymbol{j}(t, \boldsymbol{x})$ the current density.

The two fields can be expressed in terms of some potential functions $\phi(t, \boldsymbol{x})$ and $\boldsymbol{A}(t, \boldsymbol{x})$ as

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{5.2a}$$
$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{5.2b}$$

However, even if we know the expression for the EM fields, there is an infinte number of potentials related to those fields. This is called gauge freedom as a change in the potentials of the form,

$$\phi \to \phi + \frac{\partial \theta}{\partial t} \tag{5.3a}$$
$$\boldsymbol{A} \to \boldsymbol{A} - \boldsymbol{\nabla}\theta \tag{5.3b}$$

where $\theta(t, \boldsymbol{x})$ is some scalar function, leaves the fields unchanged.

This freedom comes from the fact that Maxwell's equations contain 6 variables (3 for each field) but eq. (5.1) only impose 4 restrictions on them, so there are 2 degrees of freedom. This two degrees of freedom give rise to the two orthogonal polarisations of light (vertical/horizontal, left-handed/right-handed,...). However, if we express the fields in terms of the potentials eq. (5.2), they have 4 variables, too many dof for our description. There are two of them that are redundant and give rise to the gauge freedom in eq. (5.3).

The way to constrain this extra dof is to impose a gauge fixing condition. In literature, the most used ones are:

- Coulomb gauge: defined by the condition

  $$\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0 \tag{5.4}$$

  There are many properties related to this gauge fixing condition* but the most interesting one implies that

  $$\boldsymbol{k} \cdot \boldsymbol{A} = 0 \qquad \text{and} \qquad \phi = 0 \tag{5.5}$$

  i.e. the wave-vector must be orthogonal to the potentials and, indeed, to the fields. Thus, the Coulomb gauge only allows two transverse polarisation modes.

- Lorentz gauge: this is a covariant extension of the previous which imposes that

  $$\partial_\mu A^\mu = 0 \tag{5.6}$$

  This condition give rise to wave propagating at the speed of light.

*Just take a look at the Wikipedia page: https://en.wikipedia.org/wiki/Gauge_fixing#Coulomb_gauge.

5.2 Covariant formulation

To express Maxwell's equations in covariant form, we introduce the antisymmetric field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{5.7}$$

In terms of $F^{\mu\nu}$ and the charge-current density $j^\mu = (\rho, \boldsymbol{j})$, the 4 equations (5.1) become

$$\partial_\nu F^{\mu\nu} = j^\mu \tag{5.8a}$$

$$\partial^{(\lambda} F^{\mu\nu)} = 0 \tag{5.8b}$$

The symbol $(\lambda\mu\nu)$ is used to indicate all the possible cyclic symmetric combinations of the indices $\lambda\mu\nu$.

From the antisymmetry of the EM tensor, taking the derivative with respect to μ in the first equation in eq. (5.8) gives

$$\partial_\mu j^\mu = \partial_\mu \partial_\nu F^{\mu\nu} = 0 \tag{5.9}$$

in consistency with the charge current conservation of the electromagnetic theory.

An easier expression of $F^{\mu\nu}$ is given in terms of the 4-potential A^μ as

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu = \partial^{[\nu} A^{\mu]} \tag{5.10}$$

In terms of the potentials, the first of the covariant Maxwell equation reads

$$\partial^2 A^\mu - \partial^\mu \partial_\nu A^\nu = j^\mu \tag{5.11}$$

Note that these two equations are also invariant under a gauge transformation by a total derivative

$$A^\mu \to A^\mu + \partial^\mu \theta \tag{5.12}$$

A Lagrangian density compatible with the field equations (5.8) is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \tag{5.13}$$

Unfortunately, this form of the Lagrangian is not suitable to quantize the field as the momentum conjugate of the first component of the potential $A^0 = \phi$ is identically zero. See this from

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F^{\mu 0}$$

and from the antisymmetry of $F^{\mu\nu}$ it follows that $\pi^0 \equiv 0$. The way this has been fix in history is by adding a term proportional to the Lorentz gauge eq. (5.6) which will eventually be made 0 but we leave it there for convenience. The form of the EM Lagrangian suitable for quantization is

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu - \frac{1}{2} (\partial_\mu A^\mu)^2 = -\frac{1}{2} (\partial_\nu A_\mu)(\partial^\nu A^\mu) - j_\mu A^\mu \tag{5.14}$$

From this Lagrangian we can safely construct the conjugate momenta

$$\pi^\mu = -\partial_0 A^\mu \tag{5.15}$$

and the equation of motion reads

$$\partial^2 A^\mu = j^\mu \tag{5.16}$$

which looks like a simplified version of what we previously obtained (some see this as a sign that we are moving in the right direction). However, eq. (5.11) and eq. (5.16) are only equivalent if the potential satisfies the Lorentz gauge

$$\partial_\mu A^\mu = 0 \tag{5.6}$$

Hence, to carry out the quantization of the theory but end up with Maxwell's equationseq. (5.8), we must in the first place quantize the theory for the general Lagrangian eq. (5.14) and after that impose the Lorentz gauge eq. (5.6) or an equivalent constraint.

5.3 Quantization

Consider a free field propagating in vacuum, so $j^\mu = 0$, the EoM eq. (5.16) reduce to the wave equation

$$\partial^2 A^\mu = 0 \tag{5.17}$$

This is equivalent to the Klein-Gordon field in the limit of massless particles, so much of the information gained when solving that type of fields can be applied here by setting $m = 0$.

Equation (5.17) enables us to expand the solution as a superposition of plane waves,

$$A^\mu = \int \frac{dk}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{k}}}} \sum_{r=0}^{3} \left[\epsilon_r^\mu(\boldsymbol{k}) a_r(\boldsymbol{k}) e^{-ikx} + \epsilon_r^{*\mu}(\boldsymbol{k}) a_r^\dagger(\boldsymbol{k}) e^{ikx}\right] = A^{+\mu}(x) + A^{-\mu}(x) \tag{5.18}$$

The sum is taken over the 4 possible degrees of freedom for the polarization represented by the 4-vectors $\epsilon_r^\mu(\boldsymbol{k})$. Although this theory give us 4 polarization modes, only two are physical (seen in nature), those are the transverse polarization, the other two correspond to longitudinal polarization (in the direction of motion) and a scalar polarization (in the time component). For now, the set $\{\epsilon_r^\mu(\boldsymbol{k})\}$ forms a complete set of solutions and verifies the following normalisation and completeness conditions

$$\epsilon_r^\mu(\boldsymbol{k})\epsilon_{s\mu}(\boldsymbol{k}) = \eta_{rs} \tag{5.19}$$

$$\sum_r \eta^{rr} \epsilon_r^\mu(\boldsymbol{k})\epsilon_r^\nu(\boldsymbol{k}) = -\eta^{\mu\nu} \tag{5.20}$$

for $r, s = 0, \ldots, 3$ and η_{rs} being the Minkowski metric defined like in the very beginning of the notes*. If the propagation is on the direction $\boldsymbol{k}$, a specific choice of the polarization vectors is

$$\epsilon_0^\mu(\boldsymbol{k}) = n^\nu = (1, 0, 0, 0) \tag{5.21a}$$

$$\epsilon_3^\mu(\boldsymbol{k}) = (0, \boldsymbol{\epsilon}_3(\boldsymbol{k})) = (0, \boldsymbol{k}/|\boldsymbol{k}|) = \frac{k^\mu - (kn)n^\mu}{\sqrt{(kn)^2 - k^2}} \tag{5.21b}$$

that correspond to the scalar and longitudinal polarisations. The other two, the physical ones, ϵ_1, ϵ_2 are mutually orthogonal vectors of $\boldsymbol{\epsilon}_3$ $(\boldsymbol{k})$, i.e.

$$\boldsymbol{\epsilon}_r(\boldsymbol{k}) \cdot \boldsymbol{\epsilon}_s(\boldsymbol{k}) = \delta_{rs} \qquad r, s = 1, 2, 3 \tag{5.22}$$

We now apply the canonical quantisation relations like in the previous types of fields. Taking the conjugate momenta π^μ from eq. (5.15) we impose the equal time commutation relations

$$\begin{aligned} [A^\mu(t, \boldsymbol{x}), \dot{A}^\nu(t, \boldsymbol{y})] &= -i\eta^{\mu\nu}\delta(\boldsymbol{x} - \boldsymbol{y}) \\ [A^\mu(t, \boldsymbol{x}), A^\nu(t, \boldsymbol{y})] &= [\dot{A}^\mu(t, \boldsymbol{x}), \dot{A}^\nu(t, \boldsymbol{y})] = 0 \end{aligned} \tag{5.23}$$

Apart from the factor $-\eta^{\mu\nu}$, these equations are identical with the commutation relations (3.7) of four independent KG fields, and each component $A^\mu(x)$ satisfies the wave equation (5.17) for particles of mass zero.

Gupta-Bleuler formalism Unfortunately, we cannot take simply the Lorentz condition in eq. (5.6) to fix the gauge freedom as it is incompatible with the commutation relations. Take the total derivative with respect to A^μ in eq. (5.23),

$$[\partial_\mu A^\mu(x), A^\nu(y)] = -i\partial^\nu \delta(\boldsymbol{x} - \boldsymbol{y})$$

that it is not identically 0. The problem is solved by introducing the Gupta-Bleuler condition that imposes a weaker constrain

$$\partial_\mu A^{+\mu}(x) \,|\Psi\rangle = 0 \qquad , \qquad \langle\Psi| \,\partial_\mu A^{-\mu}(x) \tag{5.24}$$

*Some books, instead of using the metric, define some new variables $\xi_0 = -1$ and $\eta_i = 1$. I find more clarifying to use the metric here, instead of having to remember the definition of those variables, in any case, remember that this depends on the definition. If one prefers to use $\eta^{\alpha\beta} = \text{diag}(-+++)$ then, in this chapter, replace all $\eta_{rs} \to -\eta_{rs}$

that involves annihilation operators only. The Lorentz condition follows from this one for expectation values

$$\langle\Psi|\partial_\mu A^\mu|\Psi\rangle = \langle\Psi|\partial_\mu A^{+\mu}(x) + \partial_\mu A^{-\mu}(x)|\Psi\rangle = 0 \tag{5.25}$$

In momentum space, we obtain the conditions

$$[a_3(\boldsymbol{k}) - a_0(\boldsymbol{k})]\,|\Psi\rangle = 0 \qquad , \qquad \langle\Psi|\,[a_3^\dagger(\boldsymbol{k}) - a_0^\dagger(\boldsymbol{k})] \tag{5.26}$$

This is a constraint on the linear combinations of longitudinal and scalar photons, for each $\boldsymbol{k}$, that may be present in a state.

Fock space To gain the photon interpretation of the quantised fields, we substitute the field expansions eq. (5.18) in the commutation relations (5.23), having

$$\begin{aligned}[a_r(\boldsymbol{k}), a_s^\dagger(\boldsymbol{k})] &= -\eta_{rs}\delta(\boldsymbol{k}-\boldsymbol{k}') \\ [a_r(\boldsymbol{k}), a_s(\boldsymbol{k}')] &= [a_r^\dagger(\boldsymbol{k}), a_s^\dagger(\boldsymbol{k}')] = 0\end{aligned} \tag{5.27}$$

Observe that for $r = s = 0$, the commutator is negative and it seems like the creation and absorption operators must be interchanged. However, effecting only this change results in other difficulties. Of the several procedures available, we shall follow that due to Gupta and Bleuler.

In the Gupta-Bleuler theory, the operators $a_r(\boldsymbol{k})$, $r = 0, 1, 2, 3$ are interpreted as creation operators and $a_r^\dagger(\boldsymbol{k})$, $r = 0, 1, 2, 3$ as annihilation operators. The vaccum state is defined as the state in which there are no photons of any kind present

$$a_r(\boldsymbol{k})\,|0\rangle = 0 \qquad \forall\boldsymbol{k}, \quad r = 0, 1, 2, 3 \tag{5.28}$$

or, in position space,

$$A^{\mu+}\,|0\rangle = 0 \qquad \forall x, \quad \mu = 0, 1, 2, 3 \tag{5.29}$$

The operators $a_r^\dagger(\boldsymbol{k})$ create a photon with polarisation r and wave-vector $\boldsymbol{k}$. A general Fock state may be represented by

$$|\boldsymbol{k}_1, r_1; \boldsymbol{k}_2, r_2; \ldots; \boldsymbol{k}_n, r_n\rangle = a_{r_1}^\dagger(\boldsymbol{k}_1)a_{r_2}^\dagger(\boldsymbol{k}_2)\cdots a_{r_n}^\dagger(\boldsymbol{k}_n)\,|n\rangle \tag{5.30}$$

It follows, from the commutation relations, that photons are bosons, i.e. two photons can exist in the same state of polarisation and momentum and the state is invariant under any permutation of the particles.

For now, the theory seems give rise to nonphysical states, those with $r = 0, 3$ but we will see that this is not the case. To justify this interpretation of the operators a_r and $a_r^\dagger$, consider the Hamiltonian operator given by

$$\mathrm{H} = \int d^3x\ \mathbf{N}\left[\pi^\mu(x)A_\mu(x) - \mathcal{L}(x)\right] \tag{5.31}$$

On substituting the Lagrangian corresponding to eq. (5.14) and the Fourier expansion of the fields eq. (5.18), the Hamiltonian becomes

$$\mathrm{H} = -\int d^3k\ \omega_{\boldsymbol{k}} \sum_r \eta_{rr} a_r^\dagger(\boldsymbol{k})a_r(\boldsymbol{k}) = \int d^3k\ \omega_{\boldsymbol{k}} \sum_{r=0}^{3} N_r(\boldsymbol{k}) \tag{5.32}$$

where the sum is over all 4 polarisation and we have introduced the number operators

$$N_r(\boldsymbol{k}) = -\eta_{rr} a_r^\dagger(\boldsymbol{k})a_r(\boldsymbol{k}) \tag{5.33}$$

Despite on the minus sign in the $N_0(\boldsymbol{k})$ operator, the energy of the system is positive semi-definite ($H \geq 0$). Indeed, when calculating the expectation value of some state $|\Psi\rangle$, by virtue of the Gupta-Bleuler condition (5.26), we have

$$\langle 0|\mathcal{H}|0\rangle = \int d^3k\ \omega_{\boldsymbol{k}} \left(\sum_{r=1}^{2} \langle\Psi|N_r|\Psi\rangle + \langle 0|N_0 + N_3|0\rangle\right) = \int d^3k\ \omega_{\boldsymbol{k}} \sum_{r=1}^{2} \langle\Psi|N_r|\Psi\rangle \tag{5.34}$$

where we have used that

$$\langle 0|N_0 + N_3|0\rangle = \langle \Psi|a_3^\dagger(\boldsymbol{k})a_3(\boldsymbol{k}) - a_0^\dagger(\boldsymbol{k})a_0(\boldsymbol{k})|\Psi\rangle \overset{(5.26)}{=} \langle \Psi|a_3^\dagger(\boldsymbol{k})\left[a_3(\boldsymbol{k}) - a_0(\boldsymbol{k})\right]|\Psi\rangle = 0 \quad (5.35)$$

Thus, as a result of the subsidiary condition, in free space observable quantities will involve transverse photons only. This explains our earlier assertion that longitudinal and scalar photons are not observed as free particles. Only transverse photons are so observed, corresponding to the two degrees of freedom (for each $\boldsymbol{k}$) of the radiation field. In the covariant treatment, although they don't show up as free particles, the presence of longitudinal and scalar photons is not ruled out altogether. Of the resulting additional two degrees of freedom (for each $\boldsymbol{k}$), one is removed the condition (5.24). The other can be shown to correspond to the arbitrariness in choice of Lorentz gauge. More specifically, one can show that altering the allowed admixtures of longitudinal and scalar photons is equivalent to a gauge transformation between two potentials both of which are in Lorentz gauges.

For free fields (i.e. no charges present), it is then simplest to work in a gauge such that the vacuum is represented by the state $|0\rangle$ in which no photons of any kind are present. But the vacuum could also be described by any state containing no transverse and only allowed admixtures of scalar and longitudinal photons. This description would merely correspond to a different choice of Lorentz gauge. The situation is entirely analogous for states containing transverse photons.

However, in the presence of charges, we can no longer ignore the longitudinal and scalar photons as they play an important role as virtual particles and provide a covariant description of the Coulomb interaction.

5.4 The photon propagator

The commutation relations of the field for two general space time points gives the propagator for the field, which in this case is represented by a matrix $D^{\mu\nu}$,

$$[A^\mu(x), A^\nu(y)] = \langle 0|\mathbf{T}\left\{A^\mu(x)A^\nu(y)\right\}|0\rangle = iD^{\mu\nu}(x-y) \quad (5.36)$$

In comparison with the Klein-Gordon propagator, we can interpret $D^{\mu\nu}(x)$ as

$$D^{\mu\nu}(x) = \lim_{m\to 0}(-\eta^{\mu\nu})\Delta(x) \quad (5.37)$$

Obviously, this propagator vanishes outside the light-cone as it does the KG propagator and thus, causality is preserved.

From eqs. (3.31) and (3.35), the photon propagator reads

$$iD^{\mu\nu}(x) = -i\eta^{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2 + i\epsilon} \quad (5.38)$$

$$iD^{\mu\nu}(k) = -i\eta^{\mu\nu}\frac{1}{k^2 + i\epsilon} \quad (5.39)$$

The completeness relation eq. (5.20) gives us a relation between the polarization states and the metric. Substituting this expression in eq. (5.39) gives

$$D^{\mu\nu}(k) = \frac{1}{k^2}\left\{\sum_{r=1}^{2}\epsilon_r^\mu(\boldsymbol{k})\epsilon_s^\nu(\boldsymbol{k}) + \frac{[k^\mu - (kn)n^\mu][k^\nu - (kn)n^\nu]}{(kn)^2 - k^2} - n^\mu n^\nu\right\} \quad (5.40)$$

Here we have taken the limit $\epsilon \to 0$ for simplicity. We can interpret this terms as

$$D^{\mu\nu}(k) = D_\perp^{\mu\nu}(k) + D_\parallel^{\mu\nu}(k) + D_0^{\mu\nu}(k)$$

that correspond to the transverse polarisation, longitudinal polarisation and scalar respectively.

The first, $D_\perp^{\mu\nu}(k)$ corresponds to the interaction of charges via the transverse field. We focus our study in the second and third terms, for the non-physical polarisations. First, note that the sum of $D_\parallel^{\mu\nu}(k) + D_0^{\mu\nu}(k)$ can be expressed as

$$D_{\parallel,0}^{\mu\nu}(k) = \frac{n^\mu n^\nu}{(kn)^2 - k^2} + \frac{1}{k^2}\left[\frac{k^\mu k^\nu - (kn)(k^\mu n^\nu + k^\nu n^\mu)}{(kn)^2 - k^2}\right] = D_C^{\mu\nu}(k) + D_R^{\mu\nu}(k) \quad (5.41)$$

the reason for these labels will be given in a moment.

Let's calculate the amplitude of the exchange of a photon between two charges $j_A^\mu(x)$ and $j_B^\mu(y)$, this is given by the expression

$$A = \int d^4x \int d^4y j_A^\mu(x) D_{\mu\nu}(x-y) j_B^\nu(y) = \int d^4k j_A^\mu(-k) D_{\mu\nu}(k) j_B^\nu(k) = A_\perp + A_{\parallel,0} \tag{5.42}$$

introducing the Fourier transform of the three quantities. Now, take the part of the propagator containing only longitudinal and scalar bosons $A_{\parallel,0}$. This is given by eq. (5.41), but note that the terms proportional to k^μ or k^ν will vanish inside the expression for the amplitude by virtue of current conservation, $k_\mu j^\mu = 0$. Then, the only surviving term is $D_C^{\mu\nu}(k)$, which gives

$$A_{\parallel,0} = \int d^4k \; j_A^\mu(-k) \frac{n_\mu n_\nu}{(kn)^2 - k^2} j_B^\nu(k) = \int d^4k \; j_A^0(-k) \frac{1}{|\boldsymbol{k}|^2} j_B^0(k) = \int d^4k \; \frac{\rho_A(-k)\rho_B(k)}{|\boldsymbol{k}|^2}$$

where we have identified $j^0(k) = \rho(k)$, the charge density. Going back to coordinate space,

$$\begin{aligned} A_{\parallel,0} &= \int d^4x \int d^4y \; \rho_A(x)\rho_B(x) \int d^3k \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})}}{|\boldsymbol{k}|^2} \int dk^0 e^{-ik^0(y^0-x^0)} \\ &= \int d^4x \int d^4y \; \frac{\rho_A(x)\rho_B(x)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \delta(x^0 - y^0) \end{aligned} \tag{5.43}$$

which is nothing but Coulomb's law written in some fancy way. The delta function is very illuminating as it tells us that the Coulomb interaction between charges occurs instantaneously. We conclude that the longitudinal and scalar bosons are responsible for the interaction between charged particles although they are not physical states.

In a similar way as in the meson propagator, it is useful to visualise it in terms of Feynman diagrams. For $y^0 < x^0$, a photon (of any polarization) is created at y that propagates through space-time until it is annihilated at x. On the other hand, for $y^0 > x^0$, the propagation is pictured as a photon that goes from x (where it is created) to y (where it is annihilated).

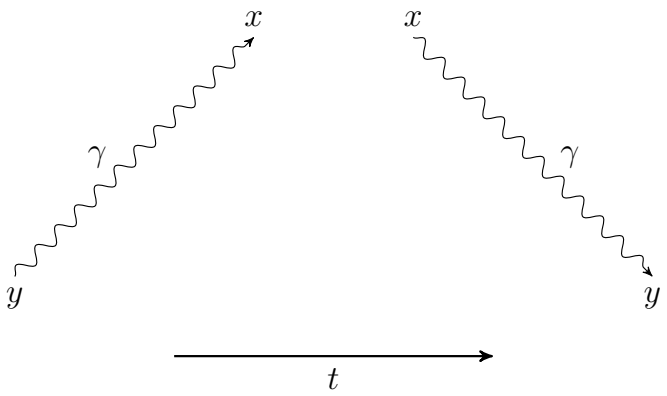


Figure 5.1: Left: $y^0 < x^0$, photon is propagated from y to x. Right: $y^0 > x^0$, photon is propagated from x to y.

In contrast to the fermion propagator, the arrows always go in the direction of time, this is a conclusion that follows from the fact that the photon is its own antiparticle.

6 The $\hat{S}$-matrix expansion

So far we have considered the free fields using the Heisenberg picture in which state vectors are constant in time and operators carry the full time dependence. We now turn to the study of the interaction and it will be more useful to work in the interaction picture [MS10], where both the state and the observables evolve with time. The complete Hamiltonian of the system in this picture can be divided into

$$\mathrm{H} = \mathrm{H}_0 + \mathrm{H}_I \tag{6.1}$$

where H_I encodes the interaction and H_0 is the part of the Hamiltonian that performs the time evolution. In this way, the Schrodinger equation reads

$$i\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle = \mathrm{H}_I(t)\left|\psi(t)\right\rangle \tag{6.2}$$

where

$$\mathrm{H}_I(t) = e^{i\mathrm{H}_0 t} H_I(t=0) e^{-i\mathrm{H}_0 t} \tag{6.3}$$

Given that the state of the system $|\psi(t=t_0)\rangle = |i\rangle$, eq. (6.2) gives the evolution of $|i\rangle$ to a later time t. In a collision, we represent the initial state much before the interaction as $|\psi(t\to-\infty)\rangle = |i\rangle$. This state contains the state of the initial particles with definite properties, like mass, charge, spin... After the interaction has occurred, an the product particles are far apart again, the final state is $|\psi(t\to\infty)\rangle = |f\rangle$. The unitary operator that relates both states is called the $\hat{S}$-matrix defined as

$$\left|\psi(\infty)\right\rangle = \hat{S}\left|\psi(-\infty)\right\rangle = \hat{S}\left|i\right\rangle \tag{6.4}$$

A collision can lead to many different final states $|f\rangle$, all of these are contained within $|\psi(\infty)\rangle$. The transition probability that after the collision, the system is in the state $|f\rangle$ is

$$P(i\to f) = |\langle f|\psi(\infty)\rangle|^2 = \left|\left\langle f|\hat{S}|i\right\rangle\right|^2 \tag{6.5}$$

If $\mathcal{F}$ is the set of all possible final states $|f\rangle$ given that the initial state was $|i\rangle$, then we expect that $\sum_f p(i\to f) = 1$. To check this, note that the set $\mathcal{F}$ must be a complete basis, i.e. $\sum_f |f\rangle\langle f| = \mathbb{I}$, then

$$\sum_f P(i\to f) = \sum_f \langle i|\hat{S}^\dagger|f\rangle\ \langle f|\hat{S}|i\rangle = \langle i|\hat{S}^\dagger\hat{S}|i\rangle = \langle i|i\rangle = 1 \tag{6.6}$$

which implies that the S-matrix must be unitary

$$\hat{S}^\dagger\hat{S} = \hat{S}\hat{S}^\dagger = \mathbb{I} \tag{6.7}$$

In order to calculate S we must solve eq. (6.2) for the initial state $|i\rangle$, this gives

$$\left|\psi(t)\right\rangle = \left|i\right\rangle - i\int_{-\infty}^{t} dt_1 \mathrm{H}_I(t_1)\left|\psi(t_1)\right\rangle \tag{6.8}$$

We encounter the state $|\psi(t)\rangle$ in both sides of the equation and usually it is impossible to solve this analytically but we can find a solution iteratively. Define $\left|\psi(t)^{(0)}\right\rangle = |i\rangle$ then

$$\left|\psi(t)^{(k+1)}\right\rangle = |i\rangle - i\int_{-\infty}^{t} dt_k \mathrm{H}_I(t_k)\left|\psi(t_1)^{(k)}\right\rangle = \sum_{j=0}^{k}(-i)^j\int_{-\infty}^{t}dt_j\int_{-\infty}^{t_j}dt_{j-1}\cdots\int_{-\infty}^{t_1}dt_0\mathrm{H}_I(t_j)\cdots\mathrm{H}_I(t_0)\left|i\right\rangle \tag{6.9}$$

Taking the limit $t\to\infty$, the S-matrix is

$$\hat{S} = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}dt_1\cdots\int_{-\infty}^{\infty}dt_n\mathbf{T}\left\{\mathrm{H}_I(t_1)\cdots\mathrm{H}(t_n)\right\} \tag{6.10}$$

where we have introduced the time-ordering operator, which orders the Hamiltonian so that later times stand to the left of earlier times and all bosons (fermions) are treated as if their corresponding fields commute (anticommute). Finally, we can express S in terms of the Hamiltonian density to implicitly show the covariance of the expression

$$\hat{S} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d^4x_1 \cdots \int_{-\infty}^{\infty} d^4x_n \mathbf{T}\left\{\mathcal{H}_I(x_1)\cdots\mathcal{H}(x_n)\right\} \tag{6.11}$$

The infinite sum can formally be written as

$$\hat{S} = \mathbf{T}\left\{\exp\left[-i\int d^4x \mathcal{H}_I(x)\right]\right\} \tag{6.12}$$

where the expansion of the exponential must be understood as the terms in eq. (6.11) and the integration is over all space-time.

6.1 Fermi's golden rule

From eq. (6.12), the transition amplitude $A(i \to f)$ is defined as the expectation value of the first-order approximation of the S-matrix

$$A(i \to f) = \langle f|S^{(1)}|i\rangle = \delta_{fi} - i\int_{-\infty}^{\infty} dt\, \langle f|\mathrm{H}_I(t)|i\rangle = \delta_{fi} - i\int_{-\infty}^{\infty} d^4x\, \langle f|\mathcal{H}_I(t)|i\rangle \tag{6.13}$$

Inserting the expression for the evolution of the interaction Hamiltonian eq. (6.3) we obtain

$$A(i \to f) = \delta_{fi} - i\,\langle f|\mathcal{H}_I(0)|i\rangle \int_{-\infty}^{\infty} d^4x e^{-i(p_i-p_f)x} = \delta_{fi} + i\hat{T}_{fi} \tag{6.14}$$

where we have defined the T-matrix as

$$\hat{T}_{fi} = \langle f|\hat{T}|i\rangle = (2\pi)^4\delta^4(p_i - p_f)\mathscr{M}(i \to f) \tag{6.15}$$

Equation (6.14) is known as Fermi's golden rule that describes the transition rate (probability of transition per unit time) from one energy eigenstate of a quantum system to a group of energy eigenstates in a continuum. This transition rate is effectively constant and is proportional to the strength of the coupling between the initial and final states of the system (described by the square of the matrix element of the perturbation) as well as the number of states per unit energy in the continuum.

The T-matrix can be formally defined as*

$$\hat{S} = \mathbf{T}\left\{e^{i\hat{T}}\right\} \approx \mathbb{I} + i\hat{T} \tag{6.16}$$

Note that it is not Hermitian, this is a result known as the *optical theorem* and is summarised as

$$i(\hat{T} - \hat{T}) = i\mathfrak{Im}\left\{\hat{T}\right\} = \hat{T}\hat{T}^\dagger \tag{6.17}$$

Note that we must work order by order in perturbation theory. But while the left handside is matrix elements, the right hand side is matrix elements squared. This means that at order 2 in some coupling constant, the left hand side must be a loop to match a tree-level calculation on the right hand side. Thus, the imaginary parts of loop amplitudes are determined completely by tree-level amplitudes. In particular, we must have the loops – the classical theory by itself,without loops, violates unitarity.

*Do not confuse the $\mathbf{T}\{-\}$ of the time ordering with the $\hat{T}$-matrix.

6.2 Wick's theorem

We must now see hot to obtain from the S-matrix expansion eq. (6.11) the transition amplitude $\langle f|\hat{S}|i\rangle$ for a particular transition $|i\rangle \to |f\rangle$ in a given order of perturbation. The Hamiltonian density $\mathcal{H}_I(t)$ involves the interaction fields, each linear in creation and annihilation operators. Hence, the expansion eq. (6.11) will describe a large number of processes. However, only certain terms will contribute to a given transition $|i\rangle \to |f\rangle$. For example, if we want to study Compton scattering, $e^-\gamma \to e^-\gamma$ where $\mathcal{H}_I(t)$ is described by the QED Hamiltonian eq. (7.5), the terms containing any creating of a e^+ can be eliminated in this calculation. They may also contain terms that create and annihilate the same particle, these particles are said to be virtual and are only present in intermediate states.

The calculation can be simplified by writing the $\hat{S}$-matrix expansion as a sum of normal products, all annihilation operators stand to the right of the creation operators. In this way, the expansion doesn't cause the emission and absorption of virtual particles. Each of this products will produce a particular transition $|i\rangle \to |f\rangle$ which can be represented by a Feynman diagram.

The method for expanding a sum of normal products which we shall now describe is due to Dyson and Wick [Wic50]. We, first of all, extend the definition of normal product made in eqs. (3.21) and (4.44) to a general formula that encloses both of them

$$\mathbf{N}\left[\Phi_1\Phi_2\cdots\Phi_N\right] = (-1)^p\Phi'_1\Phi'_2\cdots\Phi'_N \tag{6.18}$$

Here, $\Phi'_1\Phi'_2\cdots\Phi'_N$ are the operators $\Phi_1\Phi_2\cdots\Phi_N$ reordered so that all annihilation operators (positive frequency) stand on the right of creation operators (negative frequency). The exponent p is the number of interchanges of neighbouring fermionic (dirac) fields. Also, we require that the normal product is distributive

$$\mathbf{N}\left[\Phi_1\Phi_2\cdots\Phi_N + \Psi_1\Psi_2\cdots\Psi_N\right] = \mathbf{N}\left[\Phi_1\Phi_2\cdots\Phi_N\right] + \mathbf{N}\left[\Psi_1\Psi_2\cdots\Psi_N\right] \tag{6.19}$$

By this property, the normal product can thus be decomposed as a sum of normal products containing only creation or destruction operators.

From the definition of the normal product eq. (6.18) we have for two fields that

$$\Phi\Psi - \mathbf{N}\left[\Phi\Psi\right] = \begin{cases} [\Phi^+, \Psi^-] & \text{: scalar fields, bosons...} \\ \{\Phi^+, \Psi^-\} & \text{: fermions} \end{cases} \tag{6.20}$$

In any case, the result of the commutator (or anticommutator) does not involve any creation/annihilation operator, as seen in eqs. (3.7) and (4.35). Therefore, it is a complex number, by taking its expectation value with the vacuum state we deduce the relation

$$[\Phi^+, \Psi^-] = \langle 0|[\Phi^+, \Psi^-]|0\rangle = \langle 0|\Phi^+\Psi^-|0\rangle = \langle 0|\Phi\Psi|0\rangle$$

and the same happens for the anticommutator. Thus, eq. (6.20) can be written as

$$\Phi\Psi = \mathbf{N}\left[\Phi\Psi\right] + \langle 0|\Phi\Psi|0\rangle \tag{6.21}$$

In the end, we are interested in the time ordering as it is what appears in eq. (6.11). Let's see what happens when we take the time ordering of a normal product

$$\mathbf{T}\left\{\mathbf{N}\left[\Phi\Psi\right]\right\} = \theta(t_\Phi - t_\Psi)\mathbf{N}\left[\Phi\Psi\right] \pm \theta(t_\Psi - t_\Phi)\mathbf{N}\left[\Phi\Psi\right] = \left[\theta(t_A - t_B) + \theta(t_B - t_A)\right]\mathbf{N}\left[\Phi\Psi\right] = \mathbf{N}\left[\Phi\Psi\right] \tag{6.22}$$

In the first equality, the $\pm$ sign reffers to bosons/fermions (see eqs. (3.31) and (4.50)) while in the second, it has been used that $\mathbf{N}\left[\Phi\Psi\right] = \pm\mathbf{N}\left[\Phi\Psi\right]$ (also $+$ for bosons and $-$ for fermions). The two signs cancel and we are left with a plus sign in either case. This result is called **Rule C'** in [Wic50]: if two or more fields are labelled with the same time, the time ordering doesn't change the order.

Taking the time ordering of eq. (6.21) leads to

$$\mathbf{T}\left\{\Phi(x_1)\Psi(x_2)\right\} = \mathbf{N}\left[\Phi(x_1)\Psi(x_2)\right] + \langle 0|\mathbf{T}\left\{\Phi(x_1)\Psi(x_2)\right\}|0\rangle \tag{6.23}$$

For convenience, the last terms is written as

$$\overbracket{\Phi\Psi} = \langle 0|\mathbf{T}\{\Phi(x_1)\Psi(x_2)\}|0\rangle \tag{6.24}$$

which is read as, the contraction of $\Phi(x_1)$ and $\Psi(x_2)$. Being a vacuum expectation value it will vanish unless one of the operators creates a particle which the other absorbs. The non-vanishing contractions are just the Feynman propagators:

$$\overbracket{\phi(x_1)\phi^\dagger}(x_2) = i\Delta(x_1 - x_2) \tag{3.31}$$

$$\overbracket{\psi_\alpha(x_1)\psi^\dagger_\beta}(x_2) = iS_{\alpha\beta}(x_1 - x_2) \tag{4.49}$$

$$\overbracket{A^\mu(x_1)A^\nu}(x_2) = iD^{\mu\nu}(x_1 - x_2) \tag{7.21}$$

Contractions with independent fields or of the same type are identically 0.

Teorema 3 (Wick)**.** *A time product can be into a sum of normal products as*

$$\begin{aligned}\boldsymbol{T}\{\Phi_1\Phi_2\cdots\Phi_N\} = \boldsymbol{N}[\Phi_1\Phi_2\cdots\Phi_N] \\ + \sum_{i\neq j=0}^{N} \overbracket{\Phi_i\Phi_j}\boldsymbol{N}\left[\Phi_1\overset{\vee i,j}{\cdots}\Phi_N\right] \\ + \sum_{i\neq j=0}^{N}\sum_{k\neq l=0}^{N} \overbracket{\Phi_i\Phi_j}\overbracket{\Phi_k\Phi_l}\boldsymbol{N}\left[\Phi_1\overset{\vee i,j,k,l}{\cdots}\Phi_N\right] \\ + \cdots\end{aligned} \tag{6.25}$$

In the right hand side, the sum is made over all the possible contractions and it continues increasing the number of contractions. The symbol $\vee i, j$ denotes that Φ_i and Φ_j are excluded from the normal product. The proof of this relation can be made by induction starting from eq. (6.23).

However, looking at the Dyson expansion in eq. (6.11), the n-th term will be proportional to

$$\mathbf{T}\{\mathcal{H}_I(x_1)\mathcal{H}_I(x_2)\cdots\mathcal{H}_I(x_n)\} = \mathbf{T}\left\{\mathbf{N}\left[\Phi_1\Phi_2\cdots\right]_{x_1}\mathbf{N}\left[\Phi_1\Phi_2\cdots\right]_{x_2}\cdots\mathbf{N}\left[\Phi_1\Phi_2\cdots\right]_{x_n}\right\}$$

which looks familiar to eq. (6.25) except that the fields are the same but evaluated at different times. Wick extended his theorem to include this type of time ordered products, he concluded that

$$\mathbf{T}\left\{\mathbf{N}\left[\Phi_1\Phi_2\cdots\right]_{x_1}\mathbf{N}\left[\Phi_1\Phi_2\cdots\right]_{x_2}\cdots\mathbf{N}\left[\Phi_1\Phi_2\cdots\right]_{x_n}\right\} = \mathbf{T}\left\{(\Phi_1\Phi_2\cdots)_{x_1}(\Phi_1\Phi_2\cdots)_{x_2}\cdots(\Phi_1\Phi_2\cdots)_{x_n}\right\} \tag{6.26}$$

where all equal-time contractions are excluded from the sum!

Equations (6.25) and (6.26) represent the desired result, enabling us to expand each term in the $\hat{S}$-matrix expansion (6.11) into a sum of generalised normal products. Each of these normal products corresponds to a definite process, characterised by the operators not contracted which absorb and create the particles present in the initial and final states respectively. The non-vanishing contractions which occur in these generalised normal products are the Feynman propagators corresponding to virtual particles being emitted and reabsorbed in intermediate states.

6.3 The Cluster Decomposition

Let's remark a consequence of Wick theorem: any contraction of two fields at equal times vanish; this is nothing but part of a more general principle known as the Cluster Decomposition [Wei95].

Principle 1. *If multi-particle processes $\alpha_1 \to \beta_2, \ldots, \alpha_N \to \beta_N$ are studied in N very distant laboratories, then the $\hat{S}$-matrix of the overall process factorises, this is*

$$S_{\{\alpha_i\}\to\{\beta_j\}} = \sum(-1)^p S^C_{\alpha_1\beta_1}S^C_{\alpha_2\beta_2}\cdots \tag{6.27}$$

if for all of the particles in states α_i and β_i, with $i \neq j$, are at great spatial distance from α_j and β_j.

The sum in eq. (6.27) is over all different ways of partitioning the particles in the state α into clusters $\{\alpha_1, \alpha_2, \dots\}$, and likewise, a sum over all ways of partitioning the particles in state β into clusters $\{\beta_1, \beta_2, \dots\}$; not counting as different those that merely arrange particles withing a given cluster or permute whole clusters. The exponent p counts the number of fermion interchanges. The superscript C in the decomposition accounts for *connected*.

In simpler words, this principle says that the probabilities for two experiments done at the same time far appart cannot depend on each other.

For example, if the states α and β contain just a single particle, then the only term in the sum of eq. (6.27) is

$$S_{\alpha\to\beta} = S^C_{\alpha\to\beta} = \delta_{\alpha\beta}$$

The delta function must be interpreted as a product of Dirac and Kronecker deltas for each of the properties that the labels α and β encode. This is the trivial part in the $\hat{S}$ definition in eq. (6.14), equivalent to the δ_{ij}, for a process in which one particle doesn't interact with anything else.

It is convenient to reexpress this in momentum space. For a scattering of particles with momentum $\{p_i\}$ into $\{p'_j\}$, one can show that the connected parts of the $\hat{S}$-matrix should be of the form

$$S^C_{p_1 p_2 \cdots \to p'_1 p'_2} = \delta^4(p_1 + p_2 + \cdots - p'_1 - p'_2 - \cdots) C_{p_1 p_2 \cdots \to p'_1 p'_2} \tag{6.28}$$

where $C_{p_1 p_2 \cdots \to p'_1 p'_2}$ is just some coefficient that does not contain any delta. The only delta functions are global and impose 4-momentum conservation.

Take a look at what we have in the Dyson expansion of the $\hat{S}$-matrix in eq. (6.11). Each term of the expansion contains an ordered (time and normal) product of the Hamiltonian operator which in turn can be written in terms of creation and annihilation operators* that act on different times. This product, by the cluster decomposition principle, can be divided into a sum of connected parts

$$\int_{-\infty}^{\infty} dt_1 \cdots dt_n \mathbf{T}\left\{\mathcal{H}_I(t_1) \cdots \mathcal{H}_I(t_n)\right\} = \tag{6.29}$$

$$\sum (-1)^p \sum_{n_1 + \cdots n_v = n} \frac{n!}{n_1! \cdots n_v!} \prod_{j_1}^{v} \int_{-\infty}^{\infty} dt_{j_1} \cdots dt_{j_n} \mathbf{T}\left\{\mathcal{H}_I(t_1) \cdots \mathcal{H}_I(t_{n_j})\right\} \tag{6.30}$$

The first sum is over all possible v clusters like in eq. (6.27), the second sum is over all the possible ways of combining this particles with a factor to count equal particles of each type. Inserting this into eq. (6.11) we obtain

$$S_{\alpha\to\beta} = \sum (-1)^p \prod_{j=1}^{v} S^{C_j}_{\alpha\to\beta} \tag{6.31}$$

where

$$S^{C_j}_{\alpha\to\beta} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \cdots dt_n \mathbf{T}\left\{\mathcal{H}_I(t_1) \cdots \mathcal{H}_I(t_n)\right\} \tag{6.32}$$

We see that $S^{C_j}_{\alpha\to\beta}$ is calculated by a very simple prescription: *$S^{C_j}_{\alpha\to\beta}$ is the sum of all contributions to the $\hat{S}$-matrix that are connected, in the sense that we drop all terms in which any initial or final particle or any operator $\mathcal{H}_I(t)$ is not connected to all others by a sequence of particle creations and annihilations.* Compare this last sentence to the last paragraph in the previous section and find, if you can, the 7 differences.

*In fact, any operator $\mathcal{O}$ acting on particles can be expressed as a sum of products of creation and annihilation operators, see Section 4.2 in [Wei95] for a complete proof.

7 Quantum Electrodynamics (QED)

Quantum Electrodynamics is the theory that describes the interaction between electrons (and positrons) with photons. For now we have discussed how those particles evolve independently, the evolution is encoded in the respective Lagrangian densities eqs. (4.18) and (5.14). The QED Lagrangian will obviously contain those terms plus an extra one $\mathcal{L}_I$ that encodes the interaction of both particles.

7.1 Interaction Lagrangin

Previously, in section 4, we noted that the Dirac Lagrangian was invariant under phase transformation of the form

$$\psi \longrightarrow e^{i\alpha}\psi$$

and this in fact, for $\alpha = e$ (the electron charge), give us the conservation of charge theorem. To these types of transformations that do not depend on the position are named global phase transformation. There is, however, a more general one that considers space-time dependent parameter, those are local phase transformation,

$$\psi \longrightarrow e^{ie\theta(x)}\psi$$

where $\theta(x)$ is an arbitrary function and e the charge of the electron (introduced for convenience).

When we consider this type of change, the Dirac Lagrangian (4.14) is no longer invariant. The mass term remains intact, but because the derivative ∂_μ does not commute with $e^{ie\theta(x)}$, the kinetic terms suffers a change

$$i\overline{\psi}\gamma^\mu\partial_\mu\psi \longrightarrow i\overline{\psi}\gamma^\mu\partial_\mu\psi - e(\partial_\mu\theta)\overline{\psi}\psi \tag{7.1}$$

We require the Lagrangian to be invariant under local phase transformations. Thus, we may have to generalise the concept of derivative ∂_μ to covariant derivative D_μ in such a way that

$$D_\mu\psi \longrightarrow e^{ie\theta(x)}D_\mu\psi$$

It is defined as

$$D_\mu \equiv \partial_\mu - ieA_\mu \tag{7.2}$$

where A_μ is the electromagnetic potential. This is an ansaltz and there is no proof for this, it is just constructed using the minimal terms to obtain the desired result.

With the covariant derivative, the Dirac Lagrangian, with the replacement $\partial_\mu \to D_\mu$, acquires a new term

$$\mathcal{L} = \overline{\psi}(i\not{D} - m)\psi = \overline{\psi}(i\not{\partial} - m)\psi + eA_\mu\overline{\psi}\gamma^\mu\psi \tag{7.3}$$

Comparing the new piece of the Lagrangian with the term of the EM Lagrangian (see eq. (5.14)) that contains the charge, we can make the identification of

$$j^\mu = -e\overline{\psi}\gamma^\mu\psi \tag{7.4}$$

as the electric current-charge vector.

If we now add the Dirac Lagrangian (eq. (4.18)) and the EM Lagrangian (eq. (5.14)) we obtain the Lagrangian for the QED theory

$$\begin{aligned}\mathcal{L}_{QED} &= \overline{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 + eA_\mu\overline{\psi}\gamma^\mu\psi \\ &= \overline{\psi}(i\not{D} - m)\psi - \frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu)\end{aligned} \tag{7.5}$$

That can be divided into the three parts

$$\mathcal{L}_{Dirac} = \overline{\psi}(i\not{\partial} - m)\psi \tag{7.6a}$$

$$\mathcal{L}_{EM} = \frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu) \tag{7.6b}$$

$$\mathcal{L}_I = -j^\mu A_\mu = eA_\mu\overline{\psi}\gamma^\mu\psi \tag{7.6c}$$

This last part creates all the physics, all the interactions between photons, electrons and positrons.

We should calculate the expression for the interaction Hamiltonian, as this is what we will need to evaluate the $\hat{S}$-matrix element. This is easily seen to be

$$\mathcal{H}_I = -\mathcal{L}_I = -eA_\mu \overline{\psi}\gamma^\mu \psi \tag{7.7}$$

because there is no term containing a derivative in the interaction.

7.2 Feynman rules

The so called Feynman rules, provide us a way to determine the form of the matrix element $\langle f|\hat{S}|i\rangle$, for a given initial and final state, without having to calculate everything from first principles. We will start from the expression of the $\hat{S}$ in the Dyson expansion (6.11) and look, order by order, the expression for $\hat{S}$. For instance, the n-th order element of the $\hat{S}$ is

$$\langle f|S^{(n)}|i\rangle = \frac{(-i)^n}{n!}\int_{-\infty}^{\infty} dx_1^4 \cdots \int_{-\infty}^{\infty} dx_n^4 \ \langle f|\mathbf{T}\left\{\mathcal{H}(x_1)\cdots\mathcal{H}(x_n)\right\}|i\rangle \tag{7.8}$$

First of all, let's do a quick reminder on the fields and creation/annihilation operators that we have discovered in the previously. A general field ψ_l that represent particles of type l can be represented as

$$\psi_l(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3}} \sum_r \left[u_{l,r}(\boldsymbol{p})a_{l,r}(\boldsymbol{p})e^{-ipx} + v_{l,r}(\boldsymbol{p})a^\dagger_{l,r}(\boldsymbol{p})e^{ipx}\right] \tag{7.9}$$

where r is used to represent the spin, helicity or polarization of the particle l, $u_{l,r}$ and $v_{l,r}$ are the coefficients of the transform that depend on the type of field and $a_{l,r}$ ($a^\dagger_{l,r}(\boldsymbol{p})$) is the creation (destruction) operator for a particle of type l with momentum $\boldsymbol{p}$ and characterised by r. Concretely, we write here the expression for the boson, Dirac and EM fields calculated before:

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_{\boldsymbol{p}}}} \left[a(\boldsymbol{p})e^{-ipx} + a^\dagger(\boldsymbol{p})e^{ipx}\right] = \phi^+(x) + \phi^-(x) \tag{3.8}$$

$$\psi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_{\boldsymbol{p}}}} \sum_{r=\pm} \left[c_r(\boldsymbol{p})u_r(\boldsymbol{p})e^{-ipx} + d^\dagger_r(\boldsymbol{p})v_r(\boldsymbol{p})e^{ipx}\right] = \psi^+(x) + \psi^-(x) \tag{4.33}$$

$$A^\mu = \int \frac{dk}{\sqrt{(2\pi)^3 2\omega_{\boldsymbol{k}}}} \sum_{r=0}^{3} \left[\epsilon^\mu_r(\boldsymbol{k})a_r(\boldsymbol{k})e^{-ikx} + \epsilon^{*\mu}_r(\boldsymbol{k})a^\dagger_r(\boldsymbol{k})e^{ikx}\right] = A^{+\mu}(x) + A^{-\mu}(x) \tag{5.18}$$

Also, let's recap the action of the fields on a Fock state:

$$\text{Bosons:} \begin{cases} \phi^+(x) & \text{: annihilates a boson at } x \\ \phi^-(x) & \text{: creates a boson at } x \end{cases}$$

$$\text{Fermions:} \begin{cases} \psi^+(x) & \text{: annihilates a fermion of mass } m \text{ at } x \\ \psi^- & \text{: creates an antifermion of mass } m \text{ at } x \\ \overline{\psi}^+(x) & \text{: annihilates an antifermion of mass } m \text{ at } x \\ \overline{\psi}^- & \text{: creates a fermion of mass } m \text{ at } x \end{cases}$$

$$\text{Photons:} \begin{cases} A^{+\mu}(x) & \text{: annihilates a photon at } x \\ A^{-\mu}(x) & \text{: creates a photon at } x \end{cases}$$

And in momentum space:

$$\text{Bosons:} \begin{cases} a(\boldsymbol{p}) & \text{: annihilates a boson with momentum } \boldsymbol{p} \\ a^\dagger(\boldsymbol{p}) & \text{: creates a boson with momentum } \boldsymbol{p} \end{cases}$$

$$\text{Fermions:} \begin{cases} c_r(\boldsymbol{p}) & \text{: annihilates particle with momentum } \boldsymbol{p} \text{ and helicity } r \\ d_r(\boldsymbol{p}) & \text{: annihilates antiparticle with momentum } \boldsymbol{p} \text{ and helicity } r \\ c_r^\dagger(\boldsymbol{p}) & \text{: creates particle with momentum } \boldsymbol{p} \text{ and helicity } r \\ d_r^\dagger(\boldsymbol{p}) & \text{: creates antiparticle with momentum } \boldsymbol{p} \text{ and helicity } r \end{cases}$$

$$\text{Photons:} \begin{cases} a_r(\boldsymbol{k}) & \text{: annihilates a photon with wavevector } \boldsymbol{k} \text{ and polarisation} r \\ a_r^\dagger(\boldsymbol{k}) & \text{: creates a photon with wavevector } \boldsymbol{k} \text{ and polarisation} r \end{cases}$$

It is quite important to keep the previous two tables in mind for the posterior work.

To end this summary, remember that a general Fock state $|i\rangle$ can be written as a combination of creation operators that act on the vacuum state (that which is annihilated by all destruction operators and fields) as

$$|i\rangle = |\boldsymbol{p}_1, r_1; \boldsymbol{p}_2, r_2; \ldots; \boldsymbol{p}_n, r_n\rangle = a^\dagger_{l_n,r_n}(\boldsymbol{p}_n) \cdots a^\dagger_{l_2,r_2}(\boldsymbol{p}_2) a^\dagger_{l_1,r_1}(\boldsymbol{p}_1) \, |0\rangle \tag{7.10}$$

7.2.1 In position space

With all said and done, we are ready to start calculating the form of the $\hat{S}$-matrix element $\langle f|\hat{S}|i\rangle$ in position space, representing the initial and final state as some product state like in eq. (7.10), the expectation value inside the integral becomes

$$\langle 0| \left[a_{l'_n,r'_n}(\boldsymbol{p}'_n) \cdots a_{l'_2,r'_2}(\boldsymbol{p}'_2) a_{l'_1,r'_1}(\boldsymbol{p}'_1) \mathbf{T}\left\{\mathcal{H}(x_1) \cdots \mathcal{H}(x_n)\right\} a^\dagger_{l_n,r_n}(\boldsymbol{p}_n) \cdots a^\dagger_{l_2,r_2}(\boldsymbol{p}_2) a^\dagger_{l_1,r_1}(\boldsymbol{p}_1) \right] |0\rangle$$

where $\mathcal{H}(x)$ is the interaction Hamiltonian which is nothing but a product of certain fields, and indeed, a product of other creation and annihilation operators. By virtue of Wick's theorem, the time ordering product is already normal ordered but there are still creation operators on the right that come from the initial state that can be moved to the left. In the process of moving all annihilation operators to the right, we can exclude those contributions having a term like $a_{l,r}(\boldsymbol{p})\,|0\rangle$ which vanishes by definition (likewise with its adjoint). The only contributions are those arising from the delta functions when we commute two adjoint operators (see eqs. (3.7) and (5.27) and **??**). The factors are summarised here:

(a) Pairing of an initial particle $|\boldsymbol{p}_i, r_i\rangle$ with a field $\psi_l(x)$ in $\mathcal{H}(x)$,

$$[\psi_l^\dagger(x), a^\dagger_{r_i}(\boldsymbol{p}_i)]_\mp = \frac{1}{\sqrt{(2\pi)^3}} e^{-ip_i x} u_l(\boldsymbol{p}_i) \tag{7.11a}$$

(b) Pairing of an initial antiparticle $|\boldsymbol{p}_i, r_i\rangle$ with a field adjoint $\overline{\psi}_l(x)$ in $\mathcal{H}(x)$,

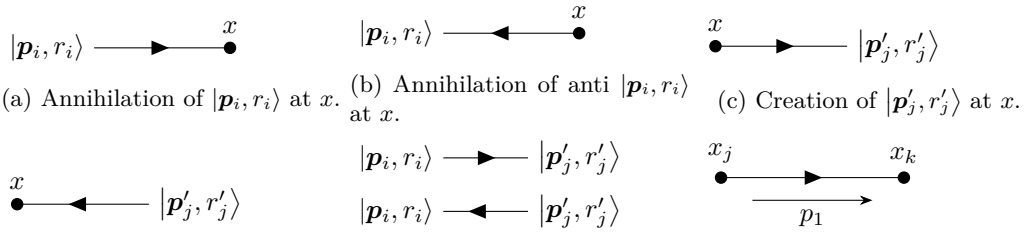
$$[\overline{\psi}_l^\dagger(x), a^\dagger_{r_i}(\boldsymbol{p}_i)]_\mp = \frac{1}{\sqrt{(2\pi)^3}} e^{-ip_i x} \overline{v}_l(\boldsymbol{p}_i) \tag{7.11b}$$

(c) Pairing of a final particle $\left|\boldsymbol{p}'_j, r'_j\right\rangle$ with a field adjoint $\overline{\psi}_l(x)$ in $\mathcal{H}(x)$,

$$[a_{r'_j}(\boldsymbol{p}'_j), \overline{\psi}_l(x)]_\mp = \frac{1}{\sqrt{(2\pi)^3}} e^{ip'_j x} \overline{u}_l(\boldsymbol{p}'_j) \tag{7.11c}$$

(d) Pairing of a final antiparticle $\left|\boldsymbol{p}'_j, r'_j\right\rangle$ with a field $\psi_l(x)$ in $\mathcal{H}(x)$,

$$[a_{r'_j}(\boldsymbol{p}'_j), \psi_l(x)]_\mp = \frac{1}{\sqrt{(2\pi)^3}} e^{ip'_j x} v_l(\boldsymbol{p}'_j) \tag{7.11d}$$



(a) Annihilation of $|\boldsymbol{p}_i, r_i\rangle$ at x.

(b) Annihilation of anti $|\boldsymbol{p}_i, r_i\rangle$ at x.

(c) Creation of $|\boldsymbol{p}'_j, r'_j\rangle$ at x.

(d) Creation of $|\boldsymbol{p}'_j, r'_j\rangle$ at x.

(e) Pairing of an initial with a final (anti)particle.

(f) Propagation of a virtual particle from x_j to x_k.

Figure 7.1: Feynman rules in position space (time flows from left to right).

(e) Pairing of a final particle $|\boldsymbol{p}'_j, r'_j\rangle$ with an initial particle $|\boldsymbol{p}_i, r_i\rangle$ (or antiparticle),

$$[a_{r'_j}(\boldsymbol{p}'_j), a^\dagger_{r_i}(\boldsymbol{p}_i)]_\mp = \delta_{r_i r'_j}\delta^3(\boldsymbol{p}_i - \boldsymbol{p}'_j) \tag{7.11e}$$

(f) Pairing of a field $\psi_l(x_j)$ with an adjoint field $\overline{\psi}_n(x_k)$,

$$[\psi_l(x_j), \overline{\psi}_m(x_k)]_\mp = -i\Delta(x_j - x_k) \tag{7.11f}$$

All of this results are better described in a diagrammatic formalism in fig. 7.1. Note that the arrows are pointing in the direction of particles moving and opposite to the direction an antiparticle moves. In cases where a particle is its own antiparticle (photon, real bosons...) the arrow should be omitted. Also, since every field or field adjoint in $\mathcal{H}$ must be paired with something else, the total number of incoming and outgoing lines at a certain vertex is equal to the total number of fields factors in $\mathcal{H}$.

To calculate the contribution to the $\hat{S}$-matrix for a given process at order n, we must carry out the following steps:

1.- Draw all Feynman diagrams containing n vertices, a line coming from the left for each initial particle and a line going to the right for each antiparticle. Joining the vertices, draw any number of intermediate steps as required to give each vertex the proper number of attached lines. Label each vertex with a spacetime coordinate x^μ.

2.- For each vertex, include a factor $(-ig_j)$, where g_j is the coupling constant multiplying the product of fields in $\mathcal{H}$. For each initial particle a factor eqs. (7.11a) and (7.11b), for each external line a factor eqs. (7.11c) and (7.11d), for each non interacting line a factor eq. (7.11e) and for each internal line connecting vertices a factor eq. (7.11f).

3.- Integrate the product of all these factor over the coordinates $x_1, x_2, \ldots$ of each vertex.

4.- Add up the results obtained from each Feynman diagram.*

I will not focus anymore on the theoretical development on the theory as it is much more complicated to explain that it really is. In Section 7 as well as in Appendix D, we will see examples of the theory were we apply the Feynman rules in position and momentum space to derive the amplitude $\mathscr{M}$ of various processes.

7.2.2 In momentum space

Usually, the Feynman rules in position space are not used as their form is much more complicated than in momentum space. This is because in position space we obtain delta factors like in eq. (7.11e) which can be integrated without problems, but also exponentials multiplied which can

*The factor $1/n!$ doesn't have to be included because for a given diagram there are $n!$ ways to order the initial and final particles which give the same amplitude. However, there are cases where extra combinatoric factors must be included, for instance, if there are m particles of the same type in some state then we obtain an extra factor of $m!$.

also be integrated giving other delta functions. Thus, going to momentum space, reduces not only the difficulty but also the length of the expressions.

I will not derive everything from first principles, but rather eunmarate the Feynam rules needed to calculate amplitudes:

1.- Draw all Feynman diagrams for a given order n (i.e. all diagrams with n vertex) and label each vertex with a off-mass shell 4-momentum q^μ.

2.- For each vertex, include a factor $(-ig_j)$ that accounts for the interaction strength as before together with a factor $(2\pi)^4\delta^4(\sum_{i\in\text{incoming}} p_i - \sum_{o\in\text{outgoing}} p'_o)$ that ensures 4-momentum conservation at each vertex.

3.- For each external line, include the corresponding factor eqs. (7.11a) and (7.11b) if the particles are in the initial state or eqs. (7.11c) and (7.11d) if the particles are in the final state; all of them without the exponential!

4.- For each internal line, add the factor $-i\Delta(\boldsymbol{q})$ where $\Delta(\boldsymbol{q})$ is the propagator of the virtual particle in momentum space.

5.- Integrate the product of all these terms over the internal 4-momentum $(2\pi)^{-4}d^4q_i$ carried by virtual particles and sum over all the possibilities for spin, helicity or polarisation.

6.- Drop the trivial term $i(2\pi)^4\delta^4(\sum_{i\in\text{initial}} p_i - \sum_{f\in\text{final}} p_f)$ in eq. (6.15) that corresponds to the $\hat{T}$ element. What is left is the amplitude $\mathscr{M}$ for that process.

7.- Add up all the amplitudes for each Feynman diagram.

In a diagram with I internal lines and V vertices, the number of independent 4-momenta that are not fixed by the delta functions is $I-(V-C)$ where C is the number of connected parts (delta functions left in the graph). Then, the number of independent loops is

$$L = I - V + C \tag{7.12}$$

which is defined as the maximum number of internal lines that can be cut without disconnecting the diagram. In particular, a tree graph is one without loops; after taking the delta functions into account there are no momentum-space integrals left for such graphs.

7.3 1st order diagrams

These are the ones generated by the first order of the $\hat{S}$-matrix,

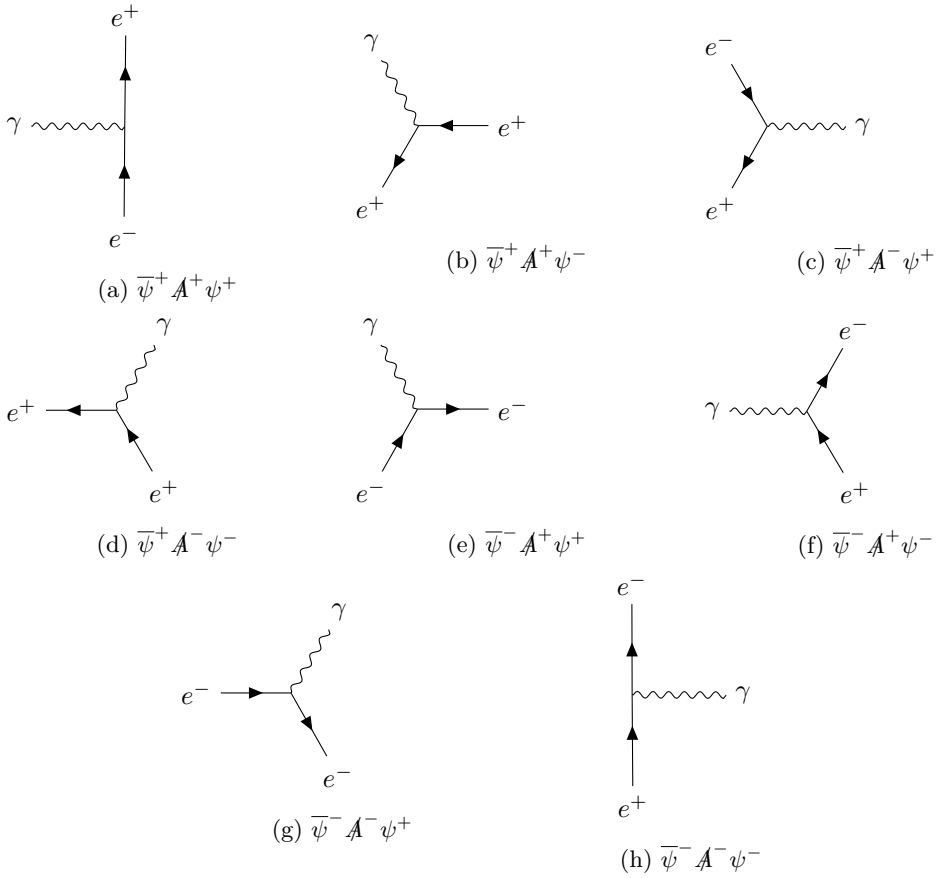
$$S^{(1)} = -i\int d^4x \mathbf{T}\left\{\mathcal{H}_I(x)\right\} = ie\int d^4x \mathbf{T}\left\{\overline{\psi}\not{A}\psi\right\} \tag{7.13}$$

where we have used the expression for the interaction Hamiltonian (7.7). The time ordering is equal to the normal ordering by Wick's theorem, since all equal time contractions are 0. Then, the first order $\hat{S}$-matrix is

$$S^{(1)} = ie\int d^4x \mathbf{N}\left[\overline{\psi}\not{A}\psi\right] = ie\int d^4x \mathbf{N}\left[(\overline{\psi}^+ + \overline{\psi}^-)(\not{A}^+ + \not{A}^-)(\psi^+ + \psi^-)\right] \tag{7.14}$$

which has 8 possible terms, each of them with one vertex, two fermion lines and one photon line. The complete set of diagrams can be seen in fig. 7.2.

All of them, despite being dynamically allowed, are kinematically forbidden. Take as an example the process $e^- \to e^-\gamma$, in the CM the energy of the initial state is m while in the second it must be greater than m because there is a photon and a moving electron. Therefore, energy and momentum are not conserved during this transitions.

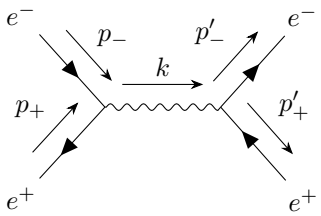


(a) $\overline{\psi}^+ \not{A}^+ \psi^+$ (b) $\overline{\psi}^+ \not{A}^+ \psi^-$ (c) $\overline{\psi}^+ \not{A}^- \psi^+$

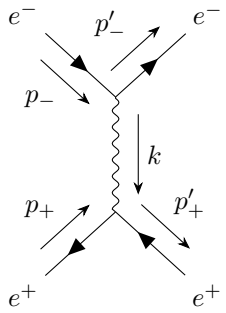
(d) $\overline{\psi}^+ \not{A}^- \psi^-$ (e) $\overline{\psi}^- \not{A}^+ \psi^+$ (f) $\overline{\psi}^- \not{A}^+ \psi^-$

(g) $\overline{\psi}^- \not{A}^- \psi^+$ (h) $\overline{\psi}^- \not{A}^- \psi^-$

Figure 7.2: First order diagrams.



(a) Annihilation



(b) Scattering

Figure 7.3: Bhabha scattering.

7.4 An example: Bhabha scattering

This makes reference to the electron-positron scattering process

$$e^- + e^+ \longrightarrow e^- + e^+$$

Both the initial and final states are composed by an electron and a positron, the terms in the $\hat{S}$-matrix expansion that generate this transition are

$$(\overline{\psi}^- \psi^-)_{x_1} (\overline{\psi}^+ \psi^+)_{x_2} \qquad , \qquad (\overline{\psi}^+ \psi^-)_{x_1} (\overline{\psi}^- \psi^+)_{x_2}$$

plus the exchange of $x_1 \leftrightarrow x_2$. The photon field is contracted an thus propagates through an internal line.

The Feynman diagrams are shown in fig. 7.3 for the first (a) and second (b) term respectively. In fact, we can obtain the second from the first, take the incoming positron in a and move it to the final state and then punt the outgoing electron in the initial state. This exchange took 3 crossing of fermion lines so we expect a minus sign to arise between the amplitudes of A and B.

From the Feynman diagrams, using the Feynman rules, we can extract the form of the amplitude, which is

$$\mathscr{M}_s = \frac{ie^2}{s} (\overline{u}' \gamma^\alpha v') (\overline{v} \gamma_\alpha u) \tag{7.15}$$

$$\mathscr{M}_t = -\frac{ie^2}{t} (\overline{v}' \gamma^\alpha v) (\overline{u}' \gamma_\alpha u) \tag{7.16}$$

where $s = (p_- + p_+)^2$ and $t = (p_- - p'_-)^2$. We have used the simplified notation $\overline{u}' = \overline{u}_{r'}(\boldsymbol{p}'_-)$, $u = u_r(\boldsymbol{p}_-)$, $\overline{v}' = \overline{v}_{s'}(\boldsymbol{p}'_+)$ and $v = v_s(\boldsymbol{p}_+)$. The total amplitude of this process is just the sum of the previous two terms.

The average amplitude over the spins of the initial and final particles is

$$\overline{|\mathscr{M}|^2} = \frac{1}{4} \sum_{r,r',s,s'=\pm} |\mathscr{M}|^2 = \frac{1}{4} \sum_{r,r',s,s'=\pm} |\mathscr{M}_s|^2 + |\mathscr{M}_t|^2 + \mathscr{M}_s \mathscr{M}_t^* + \mathscr{M}_s^* \mathscr{M}_t \tag{7.17}$$

where the factor of 4 comes from the number of total number of initial polarizations, two for the e^- and two for the e^+.

We should look at those terms one by one:

a) Take the first of the term, the absolute value can be expressed as $\mathscr{M}_s \mathscr{M}_s^*$, where the complex conjugate is equal to the conjugate transpose since $\mathscr{M}_s$ is a number, so

$$\mathscr{M}_s^* = \mathscr{M}_s^\dagger = \frac{ie^2}{s} (\overline{u}' \gamma^\alpha v')^\dagger (\overline{v} \gamma_\alpha u)^\dagger = \frac{ie^2}{s} (\overline{v}' \gamma^\alpha u') (\overline{u} \gamma_\alpha v)$$

Thus, the product gives

$$\begin{aligned}
\overline{|\mathscr{M}_s|^2} &= \frac{e^4}{4s^2} \sum_{r,r',s,s'=\pm} (\overline{u}'_a \gamma^\alpha_{ab} v'_b)(\overline{v}_c \gamma_{cd,\alpha} u_d)(\overline{v}'_e \gamma^\alpha_{ef} u'_f)(\overline{u}_g \gamma_{gh,\alpha} v_h) \\
&= \frac{e^4}{4s^2} \left((-1)^5 \sum_{r'} u'_f \overline{u}'_a \right) \left((-1)^2 \sum_{s'} v'_b \overline{v}'_e \right) \left((-1)^3 \sum_{s} v_h \overline{v}_c \right) \left(\sum_{r} u_d \overline{u}_g \right) \gamma^\alpha_{ab} \gamma_{cd,\alpha} \gamma^\alpha_{ef} \gamma_{gh,\alpha} \\
&= \frac{e^4}{4s^2} (\not{p}'_- + m)_{fa} (\not{p}'_+ - m)_{be} (\not{p}_+ - m)_{hc} (\not{p}_- + m)_{dg} \gamma^\alpha_{ab} \gamma_{cd,\alpha} \gamma^\alpha_{ef} \gamma_{gh,\alpha} \\
&= \frac{e^4}{4s^2} \operatorname{tr}\Big[(\not{p}'_- + m) \gamma^\alpha (\not{p}'_+ - m) \gamma^\beta \Big] \operatorname{tr}\Big[(\not{p}_+ + m) \gamma_\alpha (\not{p}_- - m) \gamma_\beta \Big]
\end{aligned}$$

In the next step we must use that the trace of any product of an odd number of gamma matrices vanishes, thus

$$\overline{|\mathscr{M}_s|^2} = \frac{e^4}{4s^2} \Big[\operatorname{tr}\Big(\not{p}'_- \gamma^\alpha \not{p}'_+ \gamma^\beta \Big) - 4m^2 \eta^{\alpha\beta} \Big] \Big[\operatorname{tr}\Big(\not{p}_+ \gamma_\alpha \not{p}_- \gamma_\beta \Big) - 4m^2 \eta_{\alpha\beta} \Big]$$

and using the properties of the matrices we end up with

$$\overline{|\mathscr{M}_s|^2} = \frac{8e^4}{s^2} \left\{ (p'_+ p_+)(p'_- p_-) + (p'_+ p_-)(p'_- p_+) + m^2 \left[(p'_+ p'_-) + (p_+ p_-) \right] + 2m^4 \right\} \tag{7.18}$$

b) There is no need to calculate from the beginning the scattering term since it can be obtained from the previous by the exchange $p'_- \leftrightarrow -p_+$ and $p_+ \leftrightarrow -p'_-$, thus

$$\overline{|\mathscr{M}_t|^2} = \frac{8e^4}{t^2} \left\{ (p'_+ p'_-)(p_+ p_-) + (p'_+ p_-)(p'_- p_+) - m^2 \left[(p'_- p_-) + (p_+ p'_+) \right] + 2m^4 \right\} \tag{7.19}$$

c) Finally, the next two terms can be evaluated at once by noting that $\mathscr{M}_s \mathscr{M}_t^* + \mathscr{M}_s^* \mathscr{M}_t = 2\mathfrak{Re}\left\{ \mathscr{M}_s \mathscr{M}_t^* + \mathscr{M}_s^* \mathscr{M}_t \right\}$. At the end of the day, after a lenghty calculation, we obtain

$$\mathscr{M}_s \mathscr{M}_t^* = \frac{8e^4}{st} \left\{ (p'_- p_+)(p'_+ p_-) - \frac{m^2}{2} \left[(p'_- p_-) + (p'_+ p_+) - (p_+ p_-) - (p'_- p'_+) \right] + m^4 \right\} \tag{7.20}$$

which is real and therefore the contribution of the two terms is two times this.

It is quite interesting to pause for a moment and observer the incredible symmetry of the previous three expressions...

Back in the game, the previous are easily written in terms of the Mandelstam variables

$$2p_- p_+ = 2p'_+ p'_- = s - 2m^2 \tag{7.21a}$$
$$2p_- p'_- = 2p'_+ p_+ = 2m^2 - t \tag{7.21b}$$
$$2p_- p'_+ = 2p'_+ p'_- = 2m^2 - u \tag{7.21c}$$

where we have used that $p^2 = m^2$. With this substitutions we obtain

$$\begin{aligned}
\overline{|M_s|^2} &= \frac{2e^4}{s^2} \left[t^2 + u^2 + 4m^2(s - t - u) + 8m^4 \right] \\
\overline{|M_t|^2} &= \frac{2e^4}{t^2} \left[s^2 + u^2 + 4m^2(t - s - u) + 8m^4 \right] \\
\mathscr{M}_s \mathscr{M}_t^* &= \frac{2e^4}{s^2} \left[u^2 + 4m^2(s + t - u) + 4m^4 \right]
\end{aligned}$$

8 Scattering theory

In its one-body formulation, the scattering problem is concerned with the scattering of particles by a centre of force. We consider a uniform beam of particles—whether electrons, or α-particles, or planets is irrelevant—all of the same mass and energy incident upon a fixed target (see fig. 8.1). It will be assumed that the force falls off to zero for very large distances.

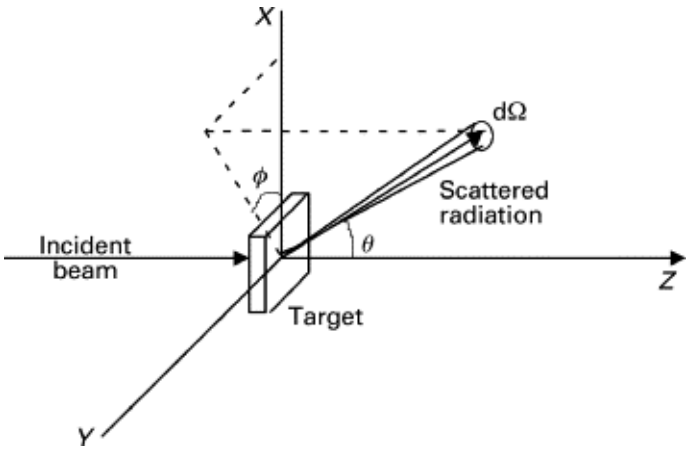


Figure 8.1: Scattering of particles by a fixed target at the origin, the resulting beam is deflected by an angle $d\Omega$.

The $\hat{S}$-matrix is the probability amplitude for the transition $i \to f$, but what does it have to do with the transition rates ans cross-sections measured in experiments? [Wei95]. The proper way to approach these problems is by studying the way experiments are actually done, using wave packets to represent particles localised far from each other before a collision, and then following the time-history of these superpositions of multi-particle states. In what follows we will instead give a quick and easy derivation of the main results.

We consider our whole system of physical particles to be enclosed in a volume V and suppose that the interaction is turned on for only a time T. The probability that a system, which is in state i before the interaction is found, afterwards, in state f is

$$P(i \to f) = \left[\frac{(2\pi)^3}{V}\right]^{N_i+N_f} |S(i \to f)|^2 \tag{8.1}$$

where N_i and N_f counts the number of particles in the initial and final state respectively. We shall define the final-state interval df in such a way that the total number of states in this range is

$$dN_f = \left[\frac{(2\pi)^3}{V}\right]^{-N_f} df \tag{8.2}$$

Hence, the total probability for the system to wind up in the range df of final states is

$$dP(i \to f) = P(i \to f)dN_f = \left[\frac{(2\pi)^3}{V}\right]^{N_i} |S(i \to f)|^2 df = \left[\frac{(2\pi)^3}{V}\right]^{N_i} \left[(2\pi)^4\delta^4(p_f - p_i)\right]^2 |\mathscr{M}(i \to f)|df \tag{8.3}$$

We will restrict to cases where no subset of the particles in the state f (other than the state itself) has precisely the same 4-momentum as some corresponding subset of the particles in the state i. In simpler words, we are eliminating the possibility of the Kronecker delta in the definition of the $\hat{S}$ amplitude eq. (6.14).

Our introduction of the box allows us to interpret the square of the delta function in eq. (8.3) as (here we have skipped some mathematical subtleties)

$$\left[(2\pi)^4\delta^4(p_f - p_i)\right]^2 = VT(2\pi)^4\delta^4(p_f - p_i)$$

In the limit of large T and V, the transition probability eq. (8.3) is simply proportional to the time T during which the interaction is acting, with the coefficient that may be interpreted as a differential transition rate

$$dw(i \to f) = \frac{dP(i \to f)}{T} = (2\pi)^3 \left[\frac{(2\pi)^3}{V}\right]^{N_i - 1} (2\pi)^4 \delta^4(p_f - p_i) |\mathscr{M}(i \to f)|^2 df \tag{8.4}$$

Here, df refers to the differential over all the phase space of the possible particles normalised as

$$df = \prod_{i=1}^{N_i} \frac{1}{2E_i} \prod_{f=1}^{N_f} \frac{d^3 p_f}{2E_f} \tag{8.5}$$

so

$$dw(i \to f) = \frac{V^{1-N_f}}{(2\pi)^{3N_f - 4}} |\mathscr{M}(i \to f)|^2 \delta^4(p_f - p_i) \Pi_{i=1}^{N_i} \frac{1}{2E_i} \Pi_{f=1}^{N_f} \frac{d^3 p_f}{2E_f} \tag{8.6}$$

This is the master formula which is used to interpret calculations of $\hat{S}$-matrix elements in terms of predictions for actual experiments.

8.1 Lifetime and decay rate

In the case where $N_i = 1$ in eq. (8.6), w is interpreted as a decay to N_f particles: $A \to 1 + 2 + \cdots + N_f$. The total decay width is defined as

$$\Gamma_A = \sum_{f \in \mathcal{F}} \Gamma(A \to f) = \frac{1}{2E_A} \int |\mathscr{M}(A \to f)|^2 (2\pi)^4 \delta^4 \left(p_A - \sum_{j=1}^{N_f} p_j\right) \prod_{j=1}^{N_f} \frac{d^3 p_j}{(2\pi)^3 2E_j} \tag{8.7}$$

the sum is taken over all the possible decays kinematically allowed for the particle A. [CDG09]

The quantity Γ_A has dimensions of E^{-1} (s^{-1}), it is related to the lifetime of the particle by

$$\tau_A = \frac{1}{\Gamma_A} \tag{8.8}$$

The quantities in the given expression are taken with respect to the CM frame and note that Γ_A is not Lorentz Invariant, i.e. it depends on the frame we are measuring, however τ_A as previously defined gives the shortest lifetime.

The quantity $\Gamma(A \to f)$, expresses the decay width for an specific process $A \to f$. If the total decay width is known, we can define the branching ratio for a given process as

$$BR(A \to f) = \frac{\Gamma(A \to f)}{\Gamma_A} \tag{8.9}$$

which gives the probability that this particular decay occurs given that we are observing the particle A. Obviously, the sum of all the branching ratios adds up to 1.

Decay into two particles For a process $A \to 1 + 2$, the rate of this process is

$$\Gamma = \frac{|\boldsymbol{p}_1^{CM}|}{32\pi^2 m_A^2} \int |\mathscr{M}(A \to 1 + 2)|^2 d\Omega \tag{8.10}$$

where $\boldsymbol{p}_1^{CM}$ is the momentum of the first scattered particle, and therefore $\boldsymbol{p}_2^{CM} = -\boldsymbol{p}_1^{CM}$ and can be calculated with the equation

$$|p_1^{CM}| = \frac{1}{2m_A} \lambda^{1/2}(m_A, m_1, m_2) \tag{8.11}$$

8.2 Cross-section

The incident beam is characterised by specifying its intensity Φ (also called flux density), which gives the number of particles crossing unit area normal to the beam in unit time. As a particle approaches the centre of force, it will be either attracted or repelled, and its orbit will deviate from the incident straight-line trajectory. After passing the centre of force, the force acting on the particle will eventually diminish so that the orbit once again approaches a straight line. In general, the final direction of motion is not the same as the incident direction, and the particle is said to be scattered. The cross section for scattering in a given direction, σ, is defined by the ratio

$$\sigma = \frac{\text{number of transitions per unit time}}{\text{incoming flux}} \tag{8.12}$$

The cross section has dimensions of area and it can be interpreted as the effective target area that sees a particle in the beam.

The numerator is given by $w(i \to f)$ while the denominator by the incoming flux $\Psi = \boldsymbol{v}/V$. In the collision of two particles, $A + B \to 1 + 2 + \cdots + N_f$, from the expression for eq. (8.6), where $N_i = 2$, we find

$$\sigma(i \to \mathcal{F}) = \frac{I}{4\lambda^{1/2}(s, m_A, m_B)} \int |\mathscr{M}(i \to f)|^2 (2\pi)^4 \delta^4 \left(p_A + p_B - \sum_{j=1}^{N_f} p_j \right) \prod_{j=1}^{N_f} \frac{d^3 p_j}{(2\pi)^3 2E_j} \tag{8.13}$$

The λ function in the denominator can also be expressed as

$$\lambda^{1/2}(s, m_A, m_B) = \sqrt{(p_A p_B)^2 - (m_A m_B)^2} = \begin{cases} \text{CM}: & \sqrt{s}\left|\boldsymbol{p}_1^{CM}\right| \\ \text{LAB}: & m_B \left|\boldsymbol{p}_1^{LAB}\right| \end{cases} \tag{8.14}$$

Also, the factor I counts the permutations between identical particles in the initial state. Ff there are n particles in the initial state, n_1 of type one, n_2 of type two... such that $n_1 + n_2 + \cdots n_v = n$ then

$$I = \frac{1}{n_1! n_2! \cdots n_v!} \tag{8.15}$$

This is a consequence of the cluster decomposition, look at eq. (6.30) and note that this factor is identical to the first coefficient in the sum. The factor of $n!$ in the numerator just cancels with the one in the definition of the $\hat{S}$-matrix and all we are left is with the value I.

The final equation for the cross section eq. (8.13) can't usually be integrated totally and the result is given in terms of the differential cross section defined as

$$\sigma = \int d\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} d\Omega \tag{8.16}$$

where $d\Omega$ is the solid angle. If the scattering is symmetric along the azimutal angle φ, then this can be integrated and express the differential cross section as differential over $\cos\theta$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \int_0^{2\pi} d\varphi = 2\pi \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \tag{8.17}$$

Two by two This is the most studied case, where the number of initial and final particles is the same, $1 + 2 \to 3 + 4$. In the CM frame, the expression for the differential cross-section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{I}{64\pi^2 s} \frac{\left|\boldsymbol{p}_f^{CM}\right|}{\left|\boldsymbol{p}_i^{CM}\right|} \left|\mathscr{M}_{1+2\to 3+4}\right|^2 \tag{8.18}$$

here $s = (E_1 + E_2)^2$ by eq. (2.12) and $\boldsymbol{p}_i$ & $\boldsymbol{p}_f$ correspond to the initial and final momentum of 1 of the particles in each state (because we analyse the situation from the CM so in modulus the momentum of both particles is the same).

However, eq. (8.18) is only valid in the CM frame. Indeed, it is possible to express the differential cross-section using only LI quantities. By using that $dt = 2\left|\boldsymbol{p}_1^{CM}\right|\left|\boldsymbol{p}_3^{CM}\right|d(\cos\theta)$, where we have used the definition of t eq. (2.3b) in the CM, it follows that [Tho11]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{I}{64\pi s\left|\boldsymbol{p}_i^{CM}\right|^2}\left|\mathscr{M}_{1+2\to3+4}\right|^2 \tag{8.19}$$

The expression in the LAB frame in terms of the solid angle is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{I}{64\pi^2\left|\boldsymbol{p}_1^{LAB}\right|m_2}\frac{\left|\boldsymbol{p}_3^{LAB}\right|^2}{\left|\boldsymbol{p}_3^{LAB}\right|(E_A+m_B)-E_3\left|\boldsymbol{p}_1^{LAB}\right|\cos\theta}\left|\mathscr{M}_{1+2\to3+4}\right|^2 \tag{8.20}$$

but if the mass of the incoming particle m_A can be neglected then this simplifies to

$$\lim_{m_A\to0}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{I}{64\pi^2}\left(\frac{1}{m_2+E_1(1-\cos\theta)}\right)^2\left|\mathscr{M}_{1+2\to3+4}\right|^2 \tag{8.21}$$

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A Natural units

It is common in the realm of the elementary particle physics to redefine units so that speed of light and Plank's constant become equal to one: $c = 1$ and $\hbar = 1$. This imposes two constraints on the three kinematical units and, therefore, leaves us a free choice for one of the three kinematical units. The units of electrical charge, also, can be and are redefined (see below). Such system of units is often referred to as Natural Units (natural for the elementary particle physics, that is). The kinematical unit of the choice is energy, E, and it is usually measured in eV (keV, MeV, GeV, TeV).

Once we fixed $c = 1$ and $\hbar = 1$, all other kinematical units can now be expressed in terms of units of energy. The relation between a unit in the IS and the equivalent in NU is

$$[\phi]_{SI} = \hbar^p c^q [\phi]_{NU} \tag{A.1}$$

The conversion factor for the most important units are given in the following table:

Quantity	SI	NU	Conversion factor
Mass	kg	E	c^{-2}
Length	m	E^{-1}	$\hbar c$
Time	s	E^{-1}	$\hbar$
Velocity	$\mathrm{m} \cdot \mathrm{s}^{-1}$	1	c
Energy	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$	E	1
Linear momentum	$\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1}$	E	c^{-1}
Angular momentum	$\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1}$	1	$\hbar$
Force	$\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}$	E^2	$(\hbar c)^{-1}$
Potential	$\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-2}$	E	c^1
Charge	C	1	$\hbar c \epsilon_0$
Current	$\mathrm{C} \cdot \mathrm{s}^{-1}$	E	$\hbar$
Fine structure	1	1	$(\hbar c \epsilon_0)^{-1}$

For the charge, it is also customary to choose $\epsilon_0 = 1$. Then, the fine structure constant in natural units has the form

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137} \tag{A.2}$$

B Noether's theorem

C Harmonic oscillator

D Toy model

Consider the Lagrangian density

$$\mathcal{L} = g\phi_A(x)\phi_B(x)\phi_C(x) \tag{D.1}$$

where $\phi_A(x)$, $\phi_B(x)$ and $\phi_C(x)$ are real scalar fields and g is the so called coupling constant with units of energy.

This Lagrangian encodes the interaction between 3 scalar particles of different mass, although this is not seen physically, the results are illustrative of the methods applied. The interaction is draw as a Feynman diagram in Figure D.1. This is the only possible way the three particles can interact, in a point were the three particles collide. This point is called a *vertex*.

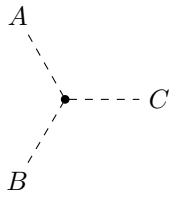

Figure D.1: Schematic interaction of the Toy Model Lagrangian.

However, the diagram in Figure D.1 does not represent a physical process. All particles going to a point or appearing from it? Indeed, we need more vertices to represent a real process. As an example, consider the scattering process $AA \to BB$. We have studied this process from a kinematical point of view in Section 2, were in Figure 2.1 the interaction is shown as a magic ball where magically the initial particles interact to give the products. Now, we know the way the particles interact because we have the Lagrangian so, the lowest order diagrams (the process with smallest number of vertices) that we can draw are shown in Figure D.2.

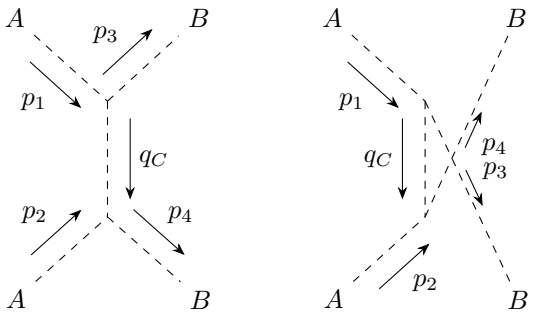

Figure D.2: Lowest order diagrams for the scattering process $AA \to BB$.

See that in each of the vertices there is a particle of type A, B and C, with C propagating as a virtual particle from one vertex to the other. We refer to the term *virtual particle* when we consider particles which are not seen in the interaction but could have appeared as a mid-step to the final products. These virtual particles exist for a brief period of time which is related to the energy of the particle by the Heisenberg relation

$$\Delta E \Delta t \sim 1 \tag{D.2}$$

Another property is that they are *off-shell*, the modulus square of its four momentum is no longer the mass, i.e. $p^2 \neq m^2$.

Let's return to the practical matter, we want to calculate $\mathscr{M}_{AA\to BB}$ at first order so we should add the amplitudes of the two possible processes showed in Figure D.2. The Feynman rules for this model are:

- For each vertex (fig. D.1) add a ig.

- For each external line (initial and final particles) multiply by a 1 (may sound stupid, but it is just to follow the standard rules).
- For each internal line multiply by the corresponding propagator for the scalar field $\phi(x)$,

$$i\Delta(q) = \frac{i}{q^2 - m^2}$$

- Impose 4-momentum conservation on each vertex, if p_{in} is the total momentum of the incoming particles and p_{out} the outgoing then

$$(2\pi)^4 \delta^4(p_{in} - p_{out})$$

- Integrate over the whole range of momentum for each internal particle considering al the above inside it

$$\int \bigstar \frac{d^4q}{(2\pi)^4}$$

- Once the integration is completed, multiply by $i(2\pi)^{-4}$ and remove the delta function concerning total 4-momentum conservation of the initial and final states, $\cancel{\delta^4(p_i - p_f)}$.
- The expression obtained after this process, if no errors have been made, is the amplitude $\mathscr{M}$.